MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 95 No. 5

MEETING OF 1935 MARCH 8

Mr. J. H. Reynolds, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Edward Howard Collinson, 16 Bath Road, Felixstowe, Suffolk ;
Henry Ernest Green, M.A., Observatory, Cambridge ; and
John Philip Manning Prentice, Redcroft, Stowmarket, Suffolk.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Henry Robert Broadhurst, White Cottage, Lime Grove, Eastcote, Ruislip, Middlesex (proposed by R. Barker) ;
C. R. Burch, Imperial College of Science, S. Kensington, S.W. 7 (proposed by F. J. M. Stratton) ;
William MacIntyre, 70 Pinner View, Harrow, Middlesex (proposed by T. E. R. Phillips) ; and
Walter Moss, Solar Physics Observatory, Cambridge (proposed by F. J. M. Stratton).

One hundred and thirty-five presents were announced as having been received since the last meeting, including :—

Barlow's Tables, 3rd edition, edited by L. J. Comrie ;
G. Lemaître and L. Bouckaert, Contributions à la théorie des effets de latitude et d'asymétrie des rayons cosmiques.
G. Lemaître, L'Univers en expansion ;
W. McCrea, Relativity Physics.

A Discussion on "The Age of the Universe" was opened by Sir James Jeans, followed by Sir Arthur Eddington and Professor E. A. Milne.

A Geophysical Meeting was held at the Society's apartments on 1935 February 22, at 16 h. 30 m., Sir Gilbert Walker in the Chair. The following papers were read and discussed :—

Dr. F. J. W. Whipple, "On an Alleged Tendency for Great Earthquakes to occur sympathetically in widely separated Regions."

Dr. H. Jeffreys, "The Surface Waves of Earthquakes."

Dr. J. de Graaff Hunter, "Regional Departures from Isostasy."

Dr. R. Stoneley, "On the Apparent Velocities of Earthquake Waves over the Surface of the Earth."

ON NEUTRAL OXYGEN AND CARBON IN NOVA HERCULIS 1934.

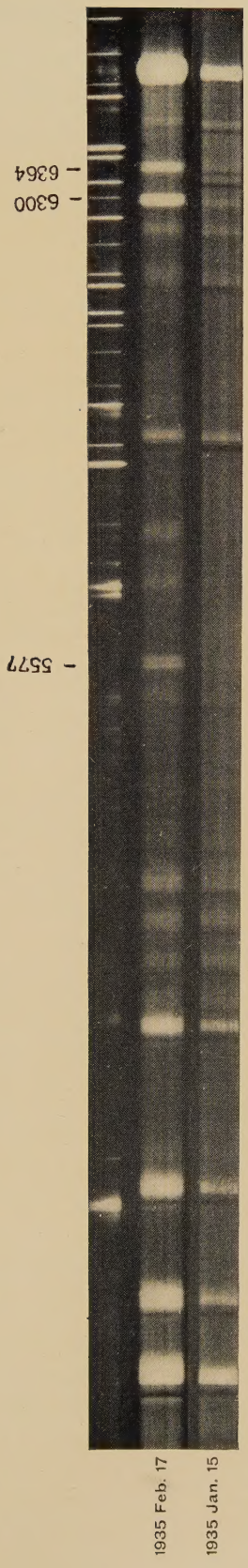
F. J. M. Stratton, D.S.O., M.A., and A. Beer, Ph.D.

1. One of the most striking changes in the spectrum of Nova Herculis 1934 during the first half of 1935 February was the strengthening of a bright band centred at 6300 A., relative to the emission bands of Fe^{+}, which are the principal features of the spectrum between $H\alpha$ and $H\beta$. An examination of the list of forbidden lines in Bowen's recent article on the gaseous nebulæ * suggested the [*O*I] line at 6300·23 A. The two accompanying [*O*I] lines, in the region of spectrum available for examination, at 6363·88 and 5577·34—the green auroral line †—were also present as bright bands in the nova, and brightened during February. The measured wave-lengths on 1935 February 17·25 are 6300·06, 6363·86 and 5578·38 A.

2. A natural step to take after making the above identifications was to examine the spectrum when the absorption lines were strongest, 1934 December 21, for ordinary absorption lines of neutral oxygen. Table I gives the result and shows that though blends may have prevented an identification of the *O*I lines previously, the lines are clearly present—at anyrate the lower members of each series ; they have the displacement to the violet (corresponding to a velocity of − 170 km./sec.) shown by the other absorption lines of the α Cygni spectrum on the same plate.

* *Ap. J.*, **81**, 1, 1935.

† McLennan, McLeod and Ruedy, *Phil. Mag.*, **6**, 558, 1928.



Bright bands of [*OI*] in the spectrum of Nova Herculis 1934.

Stratton and Beer, On Neutral Oxygen and Carbon in Nova Herculis 1934.

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TABLE I

Lines of Neutral Oxygen

λ (I.A.)	Laboratory Intensity	Multiplet	Observed λ	Velocity km./sec.	Remarks
3947·33	10	$3^5S_2^\circ - 4^5P_3$	3945·44	−150	...
3947·51	7	$3^5S_2^\circ - 4^5P_2$			
3947·61	4	$3^5S_2^\circ - 4^5P_1$			
4368·30	10	$3^3S_1^\circ - 4^3P$	4365·78	−170	Blend with 4367·68 *Ti*+ 4369·41 *Fe*+
6453·69	6	$3^5P_1 - 5^5S_2^\circ$	6453·17	...	Blend with 6456·40 *Fe*+
6454·55	7	$3^5P_2 - 5^5S_2^\circ$			
6456·07	9	$3^5P_3 - 5^5S_2^\circ$			
6155·99	7	$3^5P_1 - 4^5D^\circ$	6155·62	...	...
6156·78	8	$3^5P_2 - 4^5D^\circ$			
6158·20	10	$3^5P_3 - 4^5D^\circ$			
6140·05	...	$3^5P_1 - 4^3D^\circ$	6141·22	...	Weak and difficult
6141·07	...	$3^5P_2 - 4^3D^\circ$			
6142·38	...	$3^5P_3 - 4^3D^\circ$			
5435·16	5	$3^5P_1 - 6^5S_2^\circ$	5433·71	...	Blend with 5435·70 *Fe*+
5435·78	6	$3^5P_2 - 6^5S_2^\circ$			
5436·83	8	$3^5P_3 - 6^5S_2^\circ$			
5328·98	6	$3^5P_1 - 5^5D^\circ$	5323·86 − 5332·08	...	Hazy absorption. Blend with 5325·56 *Fe*+
5329·59	7	$3^5P_2 - 5^5D^\circ$			
5330·66	10	$3^5P_3 - 5^5D^\circ$			
5322·57	...	$3^5P_1 - 5^3D^\circ$			
5323·34	...	$3^5P_2 - 5^3D^\circ$			
5324·30	...	$3^5P_3 - 5^3D^\circ$			
4967·40	4	$3^5P_1 - 6^5D^\circ$	4965·99	...	...
4967·86	5	$3^5P_2 - 6^5D^\circ$			
4968·76	6	$3^5P_3 - 6^5D^\circ$			
4963·96	...	$3^5P_1 - 6^3D^\circ$			
4964·62	...	$3^5P_2 - 6^3D^\circ$			
4965·46	...	$3^5P_3 - 6^3D^\circ$			
5958·53	6*d*	$3^3P - 5^3D^\circ$	5956·80	...	Blend with 5960 *Si*+
5512·71	5*d*	$3^3P - 6^3D^\circ$	5509·46	−180	...
5275·08	4	$3^3P - 7^3D^\circ$	5272·37	...	Blend with 5276·00 *Fe*+ 5274·98 *Cr*+

The lines present belong to the triplet and quintet systems. The evidence for the intersystem group 3^5P to 4^3D°, 5^3D°, 6^3D° is very slight. The lines 3^5P to 6^3S°, 7^3S°, 8^3S°, 9^3S° are not found nor the lines from 3^3P to 5^5D°, 6^5D° and 6^5S°.

3. A similar examination was made for the spectra of *N*I and *C*I. Miss Moore's Multiplet Table of Astrophysical Interest gives no lines of *N*I in the range of spectrum available, while the forbidden lines 5198·5, 5200·7 would blend with Fe^+ 5197·58. In the case of *C*I there is one low-level forbidden line, at 4621·4. This, again, would blend with Fe^+ and Cr^+ lines at 4620·52, 4621·48 respectively, but no bright band is found at this wave-length. All of Miss Moore's list of lines of neutral carbon of strength 3 and upwards are found in the spectrum of the nova : they are given in Table II.

TABLE II

Lines of Neutral Carbon

λ (I.A.)	Laboratory Intensity	Multiplet	Observed λ	Velocity km./sec.	Remarks
4762·41	4	$3s^3P^\circ - 4p^3P_1$ $3s^3P_1^\circ - 4p^3P_1$	4760·34	...	Blend with 4762·78 Ti^+
4766·62	2	$3s^3P_1^\circ - 4p^3P_2$	...	...	...
4770·00	2	$3s^3P_1^\circ - 4p^3P_1$	...	...	...
4771·72	4	$3s^3P_2^\circ - 4p^3P_0$	4769·45	− 140	...
4775·89	3	$3s^3P_2^\circ - 4p^3P_1$	4773·66	− 140	...
4812·84	1	$3s^3P_0^\circ - 4p^3S_1$	4810·38	...	Blend with 4812·36 Cr^+
4817·33	1	$3s^3P_1^\circ - 4p^3S_1$	...	...	...
4826·73	1	$3s^3P_2^\circ - 4p^3S_1$	...	...	...
4932·00	5	$3s^1P_1^\circ - 4p^1P_1$	4929·30	− 160	...
5035·75	0	$3s^3P_0^\circ - 4p^3D_1$	...	...	...
5039·07	3	$3s^3P_1^\circ - 4p^3D_2$	5036·91	...	...
5041·66	3	$3s^3P_2^\circ - 4p^3D_3$			
5052·16	6	$3s^1P_1^\circ - 4p^1S_0$	5049·87	− 140	...
5380·24	8	$3s^1P_1^\circ - 4p^1D_2$	5376·95	...	Blend with 5381·03 Ti^+

4. The elements which are present singly ionised Sr^+, Ca^+, Sc^+, V^+, Cr^+, Ti^+, Fe^+, Mg^+, Si^+ have first ionisation potentials between 5·65 and 8·12: their second ionisation potentials lie between 10·98 and 16·6, while the three elements which are present in the form of neutral atoms *H*, *C* and *O* have ionisation potentials 13·54, 11·20 and 13·56 respectively. As the ultra-violet continuous spectrum of the nova is very strong, Saha's theory would suggest that it is the high density of the atmosphere which has so far prevented the next stage in the ionisation from taking place.*

We should have expected low density as a contributing factor to the appearance of the forbidden lines of *O*I, but these lines have already been obtained in the laboratory † at a pressure between 1 and 2 cm. of mercury, so that the argument for high density may stand.

* Plates taken in the first week of March 1935 show the emergence of the B-type spectrum in absorption.

† Hopfield, *Physical Review*, **37**, 160, 1931.

5. The forbidden lines of *O*I have no absorption wings on the violet side but share with the bright Fe^+ bands a common structure : two bright maxima almost equally displaced to the red and violet of the undisplaced positions of the lines are separated by an interval of 0·0019 λ ; if the displacements are due to velocity the two maxima indicate movements of matter out from the star in opposite directions with component velocities in the line of sight of $\pm$ 280 km./sec. This corresponds to the velocity of approach given by the absorption lines of Fe^+ during the first outburst on 1934 December 15, and again during most of 1935 January. The contours of the bright bands with these maxima cannot be explained by the hypothesis of expanding *shells* alone ; the presence of bright maxima on opposite sides of the undisplaced position of a line suggests an explanation in the form of streams or jets with equal velocities in opposite directions. This is a not uncommon phenomenon in novæ.

6. It is of interest to note that the lines of [*O*I] found in Nova Herculis are the exact analogues of the lines of [*O*III] so well known in nebulæ and novæ. The levels involved in the transitions—all odd levels—are

	*O*I		*O*III
$(2s)^2(2p)^4\ ^3P_2$	0·0	$(2s)^2(2p)^2\ ^3P_1$	113·6
$(2s)^2(2p)^4\ ^3P_1$	158·17	$(2s)^2(2p)^2\ ^3P_2$	306·6
$(2s)^2(2p)^4\ ^1D_2$	15867·8	$(2s)^2(2p)^2\ ^1D_2$	20273·7
$(2s)^2(2p)^4\ ^1S_0$	33792·8	$(2s)^2(2p)^2\ ^1S_0$	43186·2

and the lines are

[*O*I]		[*O*III]	
5577·3	$^1D_2 - {}^1S_0$	4363·2	$^1D_2 - {}^1S_0$
6300·2	$^3P_2 - {}^1D_2$	5007·6	$^3P_2 - {}^1D_2$
6363·9	$^3P_1 - {}^1D_2$	4959·5	$^3P_1 - {}^1D_2$

In nebulæ the lines due to the transitions from the metastable level $^1D_2$ to the low levels $^3P_2$, $^3P_1$ are stronger than the line from the higher metastable level $^1S_0$: in the aurora and the light of the night sky the line 5577 from the upper level $^1S_0$ is stronger than either of the lines 6300, 6364 from the level $^1D_2$. The case of the nova seems to be intermediate, for both in Nova Herculis 1934 and in Nova Ophiuchi 1919, where these lines were found by Dr. Wright,* 5577 is approximately as strong as 6364, though it may vary from day to day. The line 6300 appears to be about three times as intense as 6364, which agrees with Wright's estimate for this pair in nebulæ.

Grotrian has given an explanation † of the similar difference in behaviour of the three lines of [*O*III] and of [*O*I] in nebulæ and novæ which has been noted in the past, 4363 like 5577 being relatively stronger in novæ than in nebulæ. His explanation in terms of higher excitation in the novæ than in the nebulæ may hold in the present case, but it is difficult to reconcile with

* *Lick Obs. Pub.*, 14, 12, 1920.

† *Zeit. f. Phys.*, 60, 302, 1930; *Zeit. f. Ast.*, 2, 78, 1930.

the conditions in our upper atmosphere, where the line 5577 is relatively stronger still.

7. *Summary.*—Three forbidden lines of *O*I at 5577, 6300 and 6364 are present as emission bands in the spectrum of Nova Herculis 1934 and have strengthened considerably during 1935 February. They show the same structure—two bright maxima on opposite sides of the undisplaced line—as do bright bands of Fe^{+}. The displaced α Cygni absorption spectrum on 1934 December 21 shows good evidence for *C*I and fair evidence for *O*I. No evidence for *N*I is available in the range, $H\alpha$ to [K], of spectrum examined.

ON THE APPARENT AND THE REAL FORMS OF THE SOLAR CORONA.

Östen Bergstrand.

(Communicated by Prof. F. J. M. Stratton)

The form of the Sun's corona is, as a rule, rather irregular, but it may nearly always be observed that its extension is greatest in and around the equator of the Sun and least in the neighbourhood of the poles. As is known, one can trace a periodical change in this compression, in so far as it has its greatest amount at the periods of sunspot minima and is least at the maxima. It is, therefore, usual to distinguish between corona forms of "minimum type" and "maximum type."

On investigation of the real nature of these changes in the form of the corona within the 11-year cycle of solar activity one must, however, pay special attention to the fact that the apparent corona projected on the celestial sphere is formed in different ways in the direction of the equator and in the direction of the poles. In the equatorial direction the apparent corona is formed by rays which have their origin in just these solar regions. In the polar direction, on the other hand, the polar corona proper is modified by the super-stratification of the long corona rays which do not themselves belong to the polar corona but originate from low heliographic latitudes and are directed to the Earth and in the opposite direction. The outer parts of these rays belonging to the equatorial corona are projected like a veil over the polar corona proper.

To be able to investigate the influence of this veil on the intensity of the polar corona it is necessary to know approximately the law according to which the light of the corona varies with distance from the Sun's limb. This law is arrived at by micro-photometrical measurements which cover as large distances as possible from the Sun. The measurements of this kind which have been taken farthest from the Sun are my own from 1914,*

* *Études sur la distribution de la lumière dans la couronne solaire*, Stockholm, 1919.

King's and Harwood's from 1925* and von Klüber's from 1929.† In all these cases it was found that the brightness diminishes very nearly in inverse proportion to the square of the distance from the Sun's limb. In the direction of the poles this law is modified owing to the influence of the said veil. Consequently, when elaborating my observations concerning the corona of 1914 I was able to estimate this influence. It was seen that already at a distance of one solar radius from the Sun's limb the brightness of the veil was practically as great as that of the polar corona proper. At greater distances from the Sun the veil was even dominant.‡

If we denote the distance from the Sun's limb expressed in the Sun's radius as a unit by h, and if we choose as a unit for brightness that which is valid for $h = 1$ in the equatorial direction, the brightness of the equatorial corona is at a certain height h above the Sun's limb equivalent to $\frac{1}{h^2}$. The brightness of the polar corona proper at the same height is $\frac{p}{h^2}$, where p is a constant expressing the relation between the intensities of the polar corona and the equatorial corona at one and the same height above the Sun's limb. The intensity of the equatorial corona rays which are projected on the polar corona can be expressed by

$$\frac{k}{(h+n)^2} \quad (0 < n < 1),$$

where k and n are constants for every eclipse. It is a matter of determining the constants p, k and n.

For this purpose we may start from the "isophotes" of the apparent corona, *i.e.* curves of constant intensity at different distances from the Sun. If h is a certain distance from the Sun's limb in the equatorial direction and h' the polar distance in the same isophote, h and h' should for every isophote fulfil the condition

$$\frac{1}{h^2} = \frac{p}{h'^2} + \frac{k}{(h'+n)^2}.$$

In some cases, for instance for the eclipses of 1905 (Graff), 1914 (Bergstrand), 1926 and 1929 (von Klüber), the isophotes were carefully deduced by micro-photometrical measurements. For a large number of other eclipses since 1893 Ludendorff§ has deduced approximate isophotes by direct measurements on prints of corona photographs.

In a previous paper|| I have described in detail a graphic method by which—when a sufficient number of isophotes are determined by the observations—one can determine preliminarily the constants p, k and n

* *Harvard College Observatory Circular*, No. 312, 1927.

† *Zeitschrift für Astrophysik*, **3**, 157, 1931.

‡ *Loc. cit.*, p. 39.

§ *Sitzungsberichte d. Preuss. Akademie d. Wiss., Phys.-Math. Klasse*, xvi, 1928, and xvi, 1934.

|| *Arkiv f. mat., astron. och fysik*, **22**, A, No. 1 = *Meddelanden från Astronomiska Observatorium, Upsala*, Nr. 46, 1930.

for every eclipse. It was then found that with sufficient accuracy one may always put $n = 0{\cdot}5$. This means that, on an average, the corona rays, whose outer parts are projected above the polar corona and form the above-mentioned "veil," start from a heliographic latitude of about 30°. The more accurate determination of p and k can thus be obtained by means of the method of least squares according to this simple formula :

$$p + k\left(\frac{h'}{h' + 0{\cdot}5}\right)^2 - \left(\frac{h'}{h}\right)^2 = 0. \tag{1}$$

In this way I have *—starting from the isophotes determined by Ludendorff and von Klüber for 14 eclipses from 1893–1932—deduced the values of the "*coronal number*" p. This number is characteristic of the *real* form of the corona free from the influence of the veil which is projected on to the polar corona. My statement (*loc. cit.*, pp. 4 and 11) that the number k should be nearly constant for all eclipses provided the coronal structure were always completely regular is not quite correct. As in the limiting case $p = 1$ we should have theoretically $k = 0$, we ought to expect comparatively small values of k for extremely large values of p. Anyhow, the consequence of the accidental great irregularities in the coronal structure is that k may assume rather varying values for the different eclipses. On an average k has a value of about 0·6.

If we define the compression ϵ of every isophote in the apparent corona by the formula

$$\epsilon = \frac{h - h'}{1 + h'}, \tag{2}$$

we get the corresponding real compression E by putting, instead of h', $h\sqrt{p}$, and thus we have

$$E = \frac{h - h\sqrt{p}}{1 + h\sqrt{p}}. \tag{3}$$

In figs. 1 to 5 I give here a few results from those eclipses for which isophotes have been deduced up to a height of 2 Sun radii or more, viz. the eclipses of 1905 August 30, 1914 August 21, 1926 January 14, 1927 June 29 and 1929 May 9.

The values of the apparent compression ϵ of the isophotes at different distances h from the Sun's limb, obtained by Ludendorff and von Klüber from the extant observation material, are marked with a cross (×) on these figures. The lower curve shows how the observed values are represented by the compression calculated according to the formulæ (1) and (2). Considering the partly rather uncertain observation material and the irregularities in the formation of the corona the agreement may be said to be very good. This shows that the hypotheses on which formula (1) is based are rather reliable. The real compression E, free from the influence of the veil created by the equatorial rays which have been projected over the poles, and

* *Arkiv f. mat., astron. och fysik*, **25**, A, Nr. 4 = *Meddelanden från Astronomiska Observatorium, Upsala*, Nr. 60, 1934.

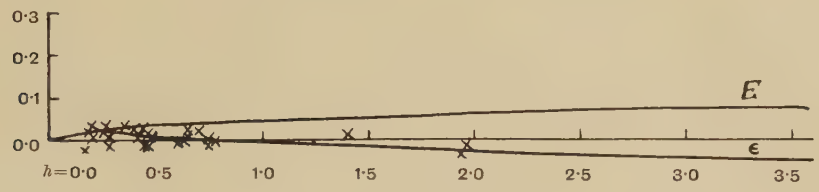

FIG. 1.—*Eclipse of* 1905 *Aug.* 30 ($p = 0{\cdot}83$; $k = 0{\cdot}37$).

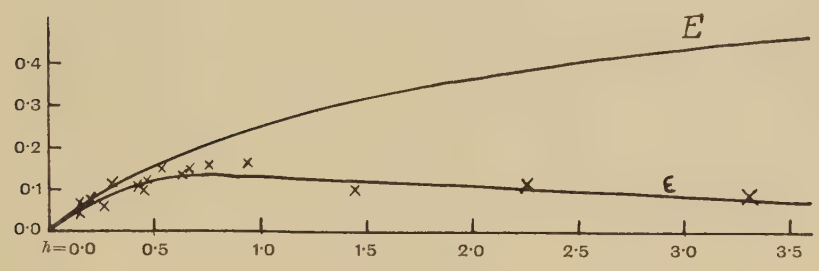

FIG. 2.—*Eclipse of* 1914 *Aug.* 21 ($p = 0{\cdot}36$; $k = 0{\cdot}62$).

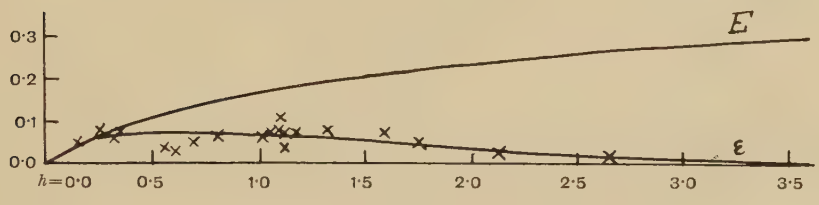

FIG. 3.—*Eclipse of* 1926 *Jan.* 14 ($p = 0{\cdot}51$; $k = 0{\cdot}62$).

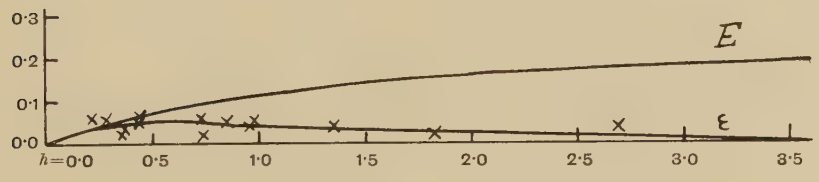

FIG. 4.—*Eclipse of* 1927 *June* 29 ($p = 0{\cdot}64$; $k = 0{\cdot}48$).

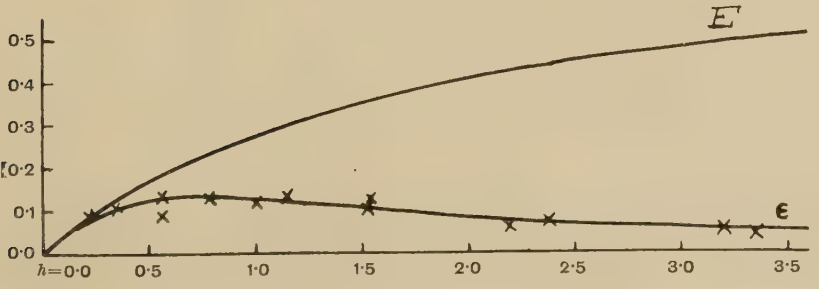

FIG. 5.—*Eclipse of* 1929 *May* 9 ($p = 0{\cdot}32$; $k = 0{\cdot}76$).

calculated according to formula (3), is shown by the upper curve. One sees that the values of ϵ and E differ very considerably from each other even at comparatively short distances from the Sun's limb, and that consequently the apparent appearance of the corona gives quite a false idea of its real form.

For a certain value of h the real compression E is only dependent upon the coronal number p, and accordingly this number lends itself provisionally (until a more accurate study of the changing forms of the corona is possible) to characterizing the form of the corona at every total eclipse of the Sun.

The coronal numbers calculated by me for different eclipses between 1893 and 1932 are given in the following table. The third column contains the phase ϕ in the 11-year cycle of the Sun's activity. This phase ϕ is defined according to Ludendorff in the following way. Let us suppose that the time for the eclipse is T. Let us further suppose that M_1 is the time of the nearest preceding sunspot maximum, and M_2 is the time of the nearest following maximum, and that m is the time of the minimum occurring between M_1 and M_2. If T lies between M_1 and m we put

$$\phi = \frac{T-m}{m-M_1}.$$

On the other hand, if T lies between m and M_2 we put

$$\phi = \frac{T-m}{M_2-m}.$$

The phase ϕ defined in this way is thus negative with a decrease of the sunspot frequency, and positive with an increase.

Eclipse	p	ϕ
1893 April 16	0·71	+0·82
1896 August 8	0·35	−0·67
1898 January 21	0·34	−0·47
1900 May 28	0·26	−0·17
1901 May 17	0·33	−0·04
1905 August 30	0·83	+0·85
1908 January 3	0·36	−0·78
1914 August 21	0·36	+0·25
1923 September 10	0·32	+0·02
1925 January 20	0·46	+0·31
1926 January 14	0·51	+0·50
1927 June 29	0·64	+0·81
1929 May 9	0·32	−0·83
1932 August 31	0·29	−0·24

In fig. 6 the coronal number p is given as a function of the phase ϕ. We find that the real form of the corona, characterized by the coronal number, does not directly depend upon the frequency of the sunspots. During the time between a sunspot maximum and a minimum this form is almost

constantly of a "minimum type" (p about $=0{\cdot}3$). From the minimum the form of the corona gradually passes from the minimum type to the "maximum type," and after the maximum it *suddenly* returns to the minimum type. This peculiar phenomenon seems to have a certain connection with the appearance of the high-latitude prominences. The mean latitude of these prominences varies in a way which closely resembles the variation of the coronal number.

Thus it seems as if the *normal* form, so to say, of the corona is strongly compressed. Only when approaching a sunspot maximum the compression decreases by the strong increase of the relative intensity of the polar corona, probably in connection with the high-latitude prominences shifting towards the immediate neighbourhood of the poles. This probable

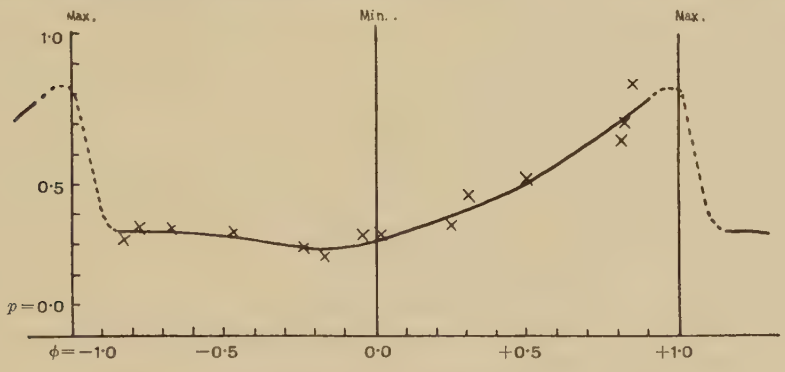

FIG. 6.—*The coronal number p as a function of the phase φ.*

connection between the variation in the form of the corona and the appearance of the high-latitude prominences has previously been pointed out by W. J. S. Lockyer.*

The discontinuity as regards the form of the corona, which as I have proved takes place shortly after a sunspot maximum, is strikingly illustrated by the interesting eclipse of 1918 June 8. Ludendorff has shown that at this eclipse the form of the corona was totally abnormal in so far as it was of the maximum type on the west side of the Sun but of the minimum type on the east side. At the same time the distribution of the sunspots on the surface of the Sun was very unsymmetrical. The coronal number for this eclipse may be estimated at about 0·22. This eclipse, which was unfortunately too scantily observed, occurred just at the critical time shortly after a sunspot maximum ($\phi = -0{\cdot}87$). It thus gives a direct indication of the fact that in this phase of the cycle of the Sun's activity an unstable condition may prevail, a condition the effects of which are entirely incalculable.

Astronomical Observatory, Upsala :
1935 *January.*

* *M.N.*, **82**, 324, 1922 ; **91**, 791, 1931.

ON THE RELATION BETWEEN THE MEAN VELOCITY OF THE STARS, THE MEAN RADIAL VELOCITY AND THE MEAN TRANSVERSE VELOCITY.

F. J. W. Whipple, M.A., Sc.D., F.Inst.P.

1. In his paper * on "Some Theorems in the Statistical Treatment of Stellar Motions," W. M. Smart has discussed the proposition, due to F. K. Edmondson, that the ratio of the mean transverse velocity of the stars to the mean radial velocity is $\pi/2$ and demonstrated that this proposition is valid under certain assumptions. Such a simple proposition should have a simple proof and it appears that the following short demonstration is adequate. The novelty in the proof is in the order of averaging. Smart averages the velocities for a small part of the sky and then for the whole sky, whereas the more effective way is to begin by averaging the component velocities corresponding to a certain absolute velocity.

It should be noted, however, that Smart's line of attack on the problem led to the discovery of certain formulæ which were only subsidiary in the proof of Edmondson's problem but which are likely to have direct applications in practical astromony.

It is common ground that, for the stars under consideration, velocity and position are to be regarded as independent. Now consider such stars as have approximately the same velocity in the same direction. The radial velocity of any one star is the absolute velocity multiplied by the cosine of the angle between the direction of motion and the radius vector, whilst the transverse velocity is the absolute velocity multiplied by the sine of the same angle. Now the average value of the cosine of the angle between a fixed axis and an arbitrary direction is $\frac{1}{2}$, whilst the average value of the sine is $\pi/4$, as may be seen by considering the values of $\int_0^{\frac{1}{2}\pi} \cos\theta \sin\theta \, d\theta$ and $\int_0^{\frac{1}{2}\pi} \sin^2\theta \, d\theta$. It follows that for the group of stars with the same absolute velocity,

$$\text{The mean radial velocity} = \text{the absolute velocity} \times \tfrac{1}{2}.$$

$$\text{The mean transverse velocity} = \text{the absolute velocity} \times \frac{\pi}{4}.$$

Hence, by averaging for all possible absolute velocities, it follows that

$$\text{The mean radial velocity} = \text{the mean absolute velocity} \times \tfrac{1}{2}$$

* *M.N.*, 95, 116, 1934.

and

$$\text{The mean transverse velocity} = \text{the mean absolute velocity} \times \frac{\pi}{4},$$

and Edmondson's rule is deduced at once:

$$\text{The mean transverse velocity} = \text{the mean radial velocity} \times \frac{\pi}{2}.$$

These propositions hold for any choice of the origin with respect to which the velocities are measured. They hold for a single star-stream as well as for the aggregate of two or more star-streams.

2. Smart discusses the case in which the stars belong to a system, in which the motions relative to a certain origin are distributed according to the Maxwellian law and the Sun has a certain velocity relative to that origin.

Let U be the velocity of the Sun and W the velocity of a star in a direction making an angle θ with the Sun's motion. The number of stars with velocities between W and $W+dW$ and with direction angles between θ and $\theta+d\theta$ is

$$2N\pi^{-\frac{1}{2}}h^3e^{-h^2W^2}W^2dW\sin\theta\,d\theta,$$

N being the total number of stars and h being related to the average velocity A by the equation,

$$A=\frac{2}{\sqrt{\pi}h}.$$

Accordingly the average velocity relative to the Sun is A_0 where

$$A_0=2\pi^{-\frac{1}{2}}h^3\int_0^{\pi}\int_0^{\infty}(W^2-2WU\cos\theta+U^2)^{\frac{1}{2}}e^{-h^2W^2}W^2dW\sin\theta\,d\theta.$$

It is to be understood that the square root has the positive sign.

On integration with respect to θ it is found that

$$A_0=\frac{2h^3}{3U\sqrt{\pi}}\int_0^{\infty}[(W+U)^3-(W\sim U)^3]e^{-h^2W^2}WdW$$

$$=\frac{4h^3}{3\sqrt{\pi}}\left[\int_0^{\infty}(3W^3+U^2W)e^{-h^2W^2}dW\right.$$

$$\left.-\frac{1}{U}\int_0^{U}(U-W)^3e^{-h^2W^2}WdW\right].$$

By evaluation of the infinite integral and by integration by parts it follows that

$$A_0=\frac{2}{\sqrt{\pi}h}\left[1+\frac{h^2}{U}\int_0^{U}(U-W)^2e^{-W^2h^2}dW\right].$$

Writing, with Smart, $hU=\tau$, $hW=x$, we obtain the equation,

$$A_0=A\phi(\tau),$$

where

$$\phi(\tau) = 1 + \frac{1}{\tau}\int_0^\tau (\tau - x)^2 e^{-x^2} dx.$$

If R_0 is the mean radial velocity relative to the Sun and T_0 the mean transverse velocity, whilst R and T refer to the velocities in the star system, it follows from § 1 that

$$R : T : A = \frac{1}{2} : \frac{\pi}{4} : 1 = R_0 : T_0 : A_0.$$

Hence we obtain Smart's results,

$$R_0 = R\phi(\tau) \quad \text{and} \quad T_0 = T\phi(\tau).$$

The function $\phi(\tau)$ has been expressed by Smart as an infinite series, viz.

$$1 + \frac{\tau^2}{1 \cdot 3 \cdot 1!} - \frac{\tau^4}{3 \cdot 5 \cdot 2!} + \frac{\tau^6}{5 \cdot 7 \cdot 3!} - \ldots$$

It may be of some interest to notice that this is the "confluent" hypergeometric series

$${}_1F_1\left[\begin{matrix} -\frac{1}{2}\,; & -\tau^2 \\ 1\frac{1}{2} & \end{matrix}\right].$$

Numerical values of $\phi(\tau)$ are set out in Table I of Smart's paper. These might be used, for example, to determine U from given estimates of R_0 and A, the appropriate formula being

$$\phi\left(\frac{2U}{\sqrt{\pi}A}\right) = 2\frac{R_0}{A}.$$

Summary.

A short proof is given of the proposition that the mean radial velocity of the stars, the mean transverse velocity and the mean absolute velocity are in the ratios $\frac{1}{2} : \frac{\pi}{4} : 1$, the velocities being with respect to any origin. Smart's formulæ connecting mean velocities with respect to the Sun and with respect to an origin travelling with a star-stream are verified.

Kew Observatory :
1935 *February* 21.

NOTE ON THE APPLICATION OF SECOND ORDER TERMS IN DIFFERENTIAL REFRACTION IN THE REDUCTION OF PHOTOGRAPHIC PLATES.

H. Spencer Jones, Sc.D., F.R.S.

The conclusions reached in the previous paper * on the second order terms in differential refraction in the measurement of photographic plates must be considered in conjunction with the methods used in the reduction of the plates. I am indebted to Sir Arthur Eddington for drawing attention to this. The following remarks are added to make the position clear, as incorrect methods of reduction have frequently been employed.

Consider a photographic plate whose centre is at O; in other words, the plate is tangential to the celestial sphere at the point O. The plate is measured in two directions, Ox, Oy, which are approximately in the directions of right ascension and declination. Suppose S' is the image of a star, whose measured co-ordinates are x, y; this image is displaced by refraction, aberration and precession. For the present purpose we need consider refraction only. Suppose S is the true position of the star, not displaced by refraction.

When we know the right ascension and declination of the plate centre, O, we can compute the standard co-ordinates of the star S. They are the co-ordinates referred to O as origin and axes in the plate in the directions of increasing right ascension and declination respectively. We denote them by ξ, η. The relationships between x, y and ξ, η can be regarded as linear, except when the second order terms in differential refraction become sufficiently large to be taken into account. It is convenient in such cases to apply small appropriate corrections to the measured positions x, y so that the relationships between x, y and ξ, η become linear. The correct formulæ for the second order terms in differential refraction required to be applied to the measured co-ordinates are Turner's formulæ, as shown in the previous paper.

The relationships between x, y and ξ, η may now be assumed to be linear and can be written in the form

$$x - \xi = ax + by + c,$$
$$y - \eta = dx + cy + f.$$

The constants c, f are corrections to the assumed plate centre, and if the co-ordinates of the plate centre have been determined initially with sufficient accuracy, c, f will be small. If they are not small (they should not exceed $2'$ for an astrographic plate covering $2^\circ \times 2^\circ$), it is necessary to recompute the standard co-ordinates using a corrected plate centre, in order to avoid errors of tilt.

The method of determining the plate centre must be considered. Taking any two convenient stars, whose images on the plate are at S_1', S_2', the

* *M.N.*, 95, 132, 1934.

position of the plate centre is interpolated between the right ascensions and declinations of these stars, using their measured co-ordinates. But it must be remembered that it is not correct to use the mean positions of the stars for this purpose ; if we do so, the plate centre will be in error by amounts kX, kY, where k denotes the constant of refraction and X, Y the co-ordinates of the zenith on the plate. The correct procedure, if this method of reduction is employed, is to correct the mean places of the stars by the amount of the refraction and to use the positions, as modified, for deriving the right ascension and declination of the plate centre, O.

The usual procedure, however, is to interpolate between the positions of the two stars, using their mean places. Under what conditions is this procedure correct? What is done in effect is to refer the positions of the stars from the true plate, tangential to the celestial sphere at O, to a hypothetical plate, tangential to the celestial sphere at another point O_1. It is obvious that O_1 will lie on the great circle passing through the zenith and O, and that O_1 is a point such that a star which would fall in that position in the absence of refraction is displaced by refraction to O. In other words, the components of OO_1 along the two axes are kX, kY.

The standard co-ordinates, computed by using the mean places of the stars and plate centre O_1, are co-ordinates in the plane tangential at O_1. We have to relate co-ordinates (x, y) measured in one plane to co-ordinates (ξ, η) measured in another plane. In order to be able to assume linear relationships between x, y and ξ, η we must now correct not only for the second order terms in differential refraction, but also allow for the effect of tilt between the plates tangential at O, O_1 respectively. The differential refraction terms are given by Turner's formulæ. The tilt terms by the usual formulæ (the components of tilt being kX, kY) are

$$dx = kX \,.\, x^2 + kY \,.\, xy,$$
$$dy = kX \,.\, xy + kY \,.\, y^2.$$

When these are added to the formulæ for the second order terms in differential refraction, viz.

$$dx = -k\{X(2+X^2)x^2 + Y(1+2X^2)xy + X(1+Y^2)y^2\},$$
$$dy = -k\{Y(1+X^2)x^2 + X(1+2Y^2)xy + Y(2+Y^2)y^2\},$$

the final expressions are

$$dx = -kX\{(1+X^2)x^2 + 2XYxy + (1+Y^2)y^2\},$$
$$dy = -kY\{(1+X^2)x^2 + 2XYxy + (1+Y^2)y^2\}.$$

These formulæ, which are equivalent to the formulæ given by Kapteyn and Baillaud for the second order terms in differential refraction, are thus a combination of differential refraction terms and tilt terms. With the usual method of reducing plates, in which the mean positions of two stars are used for determining the plate centre to be employed in the computation of the standard co-ordinates ξ, η, these formulæ give the correct expressions for the second order terms which must be applied to the measured co-ordinates, x, y, in order that the relationship between x, y and ξ, η should be a linear one.

ATMOSPHERIC CONDITIONS AT THE SITE FOR THE RADCLIFFE OBSERVATORY NEAR PRETORIA.

W. H. Steavenson and H. Knox-Shaw.

When Sir Frank Dyson and Dr. Knox-Shaw were in South Africa in 1929, attending the meeting of the British Association, they examined several possible future sites for the Radcliffe Observatory in the event of its being decided to remove it from Oxford. The one that struck them as likely to prove the most suitable for an observatory equipped with a large reflector was on the range of hills to the south-east of Pretoria, where the Municipality of that city generously offered to present to the Radcliffe Trustees a piece of land. Before making a decision it was considered important that a series of tests of the seeing should be made at this site, and Dr. Steavenson went to Pretoria at the request of the Radcliffe Trustees to undertake this.

Dr. J. A. Anderson had recently evolved a method of testing the seeing which was being employed in the search for a suitable position for the 200-inch telescope, and for the sake of comparison it seemed advantageous to make use of this method. Dr. W. S. Adams most kindly communicated to us a full account of the method, and it was adopted for the tests at Pretoria with such slight modifications as were necessitated by the different conditions. No detailed account of Dr. Anderson's method has, so far as we know, yet been published, so it would clearly not be proper to give one here. Suffice it to say that it involves the measurement of the "tremor disc," the total amplitude in seconds of arc over which the star's image oscillates owing to atmospheric unsteadiness, and in addition an estimate of the definition from the character of its diffraction pattern. For photographic work with a large telescope steadiness is of much greater importance than definition, and consequently in deriving a figure for the seeing greater weight is given to the former than to the latter. The formula provided by Dr. Anderson expresses the seeing on the Mount Wilson scale, in which the worst seeing is represented as usual by 0 and absolutely perfect seeing by 10.

The instrument used was a 6-inch refractor equatorially mounted, which was kindly lent by the Astronomer Royal. It was fitted with a double wire micrometer for measuring the tremor disc, and a power of 500 was employed. The site is four miles to the south-east of, and about 500 feet higher than, the centre of Pretoria, its altitude above sea-level being 4850 feet. An intervening hill shields it from the lights of the town. During the period of observation it was entirely unaffected by the fogs which at times covered the town. The prevailing wind is north-easterly, and Pretoria is extending industrially on its western side. The Municipality of Pretoria very kindly erected on the site a hut with sliding roof for the telescope and accommodation for the observer, and the Union Meteorologist was good enough to

provide an anemometer, thermometers, thermograph and screen. Observations were begun on 1930 February 7 and were continued until June 25. Of the 139 nights in this period 48 were overcast during the time that watch was kept. On the other 91 nights it was estimated that some work could have been done with a large telescope; over two-thirds of them were cloudless. The records in Pretoria and at the Union Observatory in Johannesburg show this to have been an unusually wet and cloudy autumn. The seeing was tested at intervals of about half an hour during the first part of the night, *i.e.* from dusk until midnight or slightly beyond, whenever the sky was sufficiently clear. Stars were observed at various altitudes. Naturally the unsteadiness was found to increase with zenith distance, but this effect was not at all marked at altitudes of over 30°. A mean value of the seeing was taken for each night from all the observations except those on stars as low as 30° in altitude. An estimate was also made of the transparency of the sky. Meteorological observations were made on the site from the beginning of March. The anemometer was mounted at a height of 20 feet above ground-level, and the direction and velocity of the wind were read periodically during the time the seeing was being observed.

TABLE I

Month	Seeing at Pretoria Site							Seeing at Mt. Wilson.
	Feb.	Mar.	Apr.	May	June	Total	Year	Mean of 6 Years
	Numbers of Days							
Nights observed	22	31	30	31	25	139	365	365
Overcast	14	11	12	7	4	48	128	82
Clear or partly so	8	20	18	24	21	91	237	283
Seeing (Mt. Wilson Scale)								
1	0	1	2	3	2	8	21	46
2	0	2	0	1	3	6	16	44
3	0	1	2	1	3	7	18	50
4	0	2	3	2	1	8	21	59
5	0	3	2	2	3	10	26	52
6	6	4	4	4	4	22	57	22
7	2	5	3	8	4	22	57	8
8	0	2	2	3	1	8	21	2

The numbers of nights in each of the months February to June having the various figures of seeing are given in Table I; also the total for the period and by proportion for the whole year; and finally, as a comparison, figures for Mount Wilson taken from the annual reports of that observatory. How far the extrapolation to the whole year at Pretoria by simple proportion from the period of observation is justifiable will be discussed later in this paper. Assuming for the moment that it gives a fair picture of normal conditions for a year, the table indicates that whereas the number of fine

nights at Pretoria is less than at Mount Wilson their quality is decidedly higher; at the former place seeing of 6 or 7 may be expected on nearly half the observing nights, while at the latter the figures 1 to 5 occur almost equally frequently and together constitute 87 per cent. of the whole. As evidence of the high quality of the definition it may be mentioned that the companion of Sirius, which is generally considered to be a difficult object with a 6-inch telescope, was nearly always easily visible when the altitude of the star exceeded 30°, and that on more than one occasion it was possible even to make satisfactory measures of its position angle and distance with the micrometer. The fifth and sixth stars in the Orion Trapezium were nearly always conspicuous objects, and on one occasion Saturn's satellite Enceladus was well seen, although the rings were at that time widely open. It often happened that to the naked eye there was no perceptible twinkling of the stars at altitudes greater than 45°.

TABLE II

Relation between Wind and Seeing at Pretoria

Wind	Nearly Calm (< 3 m.p.h.)	Northerly (N., N.E.)	Southerly (E., S.E., S., S.W.)
No. of nights	49	8	30
Mean wind velocity (m.p.h.)	...	5·0	5·7
Mean seeing	6·4	5·4	3·1

Analysis of the observations at the Pretoria site brought to light a marked dependence of the seeing on the direction of the wind, as is shown in Table II. The best seeing occurred, as we believe is usually the case, when it was calm or nearly so, but that which accompanied a wind from a northerly direction was but little inferior. All the days of bad seeing (1 to 3), except one when the wind was N.E., occurred on nights when the wind was from the east or from some southerly point. This we gather is also the experience of the observers at the Union Observatory, Johannesburg, some 30 miles away. We are indebted to the Union Astronomer for supplying us with the figures of observed seeing at Johannesburg over the period of our observations and for the observed wind directions at 21^h over a period of five years. The former show a marked correlation with the seeing as recorded at Pretoria, indicating that the bad seeing is chiefly due to high-level atmospheric disturbance and is not much influenced by local orographical movements of the air. The frequency of wind directions at Johannesburg as observed over five years indicates that the percentage of "southerly winds" (E., S.E., S., S.W., as above) is somewhat higher for the five months February to June than for the rest of the year. Consequently if we can assume, as seems reasonable in view of the experience of the Johannesburg observers, that it is these winds that are responsible for bad seeing throughout the year, the extrapolation from these five months to the whole year appears to be justifiable, so far as the quality of the seeing is concerned. As regards the number of fine nights, we are perhaps not on quite as sure ground, though

the mean amount of cloud by day at Johannesburg for the five months February to June is slightly greater than for the whole year. No figures are available to us for the amount of cloud by night, but in view of the fact that the period of observation extended over parts of the summer and winter and was more cloudy than usual for the time of year, it appears likely that the number of observing nights to be expected in a normal year is not far short of 250.

No quantitative observations of the transparency of the atmosphere were made, but it was estimated on an arbitrary scale. On only two clear nights was it classed as poor. In fact throughout the period a satisfactory standard was maintained. A series of plates, which were taken of the Milky Way with a small wide-angle lens and with a uniform exposure of two hours, confirmed the impression gained by visual observation that the transparency was sensibly uniform from night to night.

When considering a site for a large reflector the diurnal range of temperature is a factor which cannot be neglected. It is in general large on the high veld. In the town of Pretoria its annual mean value is 29° F., being about 5° greater than this in the winter months and 5° less in the summer. The range shown by the thermograph in the screen on the observatory site during the months March to June averaged 5° less than in the town. While it is of course a somewhat dangerous extrapolation to the whole year, it seems likely that the mean value of the diurnal range on the site can be taken with some confidence to be not more than 24° or 25°, and the difference between its winter and summer values to be considerably less than in the case of the town. At the Union Observatory (altitude 5925 feet) the mean diurnal range is 20°.

Summary.—So far as it is possible to determine the suitability of a place for work with a large telescope by observing with a small one, the tests carried out at the Pretoria site indicate that the seeing there is of a high order, and, assuming that Anderson's method furnishes a true comparison with the Mount Wilson scale, that it is distinctly superior to the seeing at that observatory. The number of fine nights is adequate, and they are reasonably well distributed through the year. The diurnal range of temperature is large, but the use of fused quartz or low-expansion glass for the mirrors will make this feature of less importance than it has been in the past.

We should like to express here, even though it be somewhat tardy, our deep appreciation of the kindness of the various officers of the Pretoria Municipality, and others too numerous to mention individually, who spared no pains in assisting the enterprise on every possible occasion.

PRELIMINARY VALUES OF THE VARIATION OF LATITUDE AT GREENWICH IN 1934.

(Communicated by the Astronomer Royal)

This paper contains the preliminary values of the latitude variation derived from observations made with the Cookson floating telescope.

The star places used were those derived from the final discussion of the observations for the period 1919–1926. The number of plates obtained was about the average, but owing to weather conditions there was only a small number of plates in February, November and December.

The following table gives the values of the variation (mean latitude *minus* observed latitude) :—

Date	Variation	Number of Pairs
	″	
1934·030	+·071	99
·115	+·097	80
·254	+·135	97
·344	+·167	92
·444	+·122	83
·522	+·032	83
·573	−·015	97
·636	−·062	89
·678	−·051	90
·733	−·109	86
·832	−·015	93
1935·005	+·018	77

From a curve drawn to represent the observed results as closely as possible, the following corrections, applicable to observed north polar distances, were derived :—

	″		″
1934·0	+·02	1934·5	+·05
34·1	+·08	34·6	−·03
34·2	+·12	34·7	−·07
34·3	+·14	34·8	−·05
34·4	+·13	34·9	−·02

Royal Observatory, Greenwich :
1935 *February.*

FURTHER INVESTIGATIONS OF AN APPARENT LUNAR EFFECT IN TIME DETERMINATIONS.

A. L. Loomis and H. T. Stetson.

In a previous paper * an analysis of the day-by-day time comparisons between Rugby and Annapolis for the three years 1929 to 1932 revealed an apparent lunar effect in time determinations at Greenwich and Washington. Curves were exhibited showing an apparent range of from $0^s.06$ to $0^s.12$ in $2d$, where d represented the resulting differences in longitude. While it was suggested that in view of previous studies in the variation of latitude † a small periodic movement in the Earth's crust might be a factor in producing the effect, any attempt at a discussion of the origin of the effect was postponed until there had been opportunity to make a similar study of the time comparisons between Annapolis and Bordeaux and between Bordeaux and Rugby. It will be understood of course that Annapolis, Rugby and Bordeaux are the locations of the sending stations for the broadcasting of time-signals from the U.S. Naval Observatory at Washington, the Royal Observatory at Greenwich and the Observatory at Paris.

As it appeared desirable to make an intercomparison of the results with reference to a single meridian, the data have all been reduced so as to represent longitude differences referred to the Greenwich meridian corresponding to specific hour-angles and declinations of the Moon referred to the same meridian at Greenwich Civil Time 0^h. For this reason the results published in the previous paper for an arbitrary meridian have been recalculated to present a basis for uniform comparison with the results obtained from the Rugby–Annapolis and the Bordeaux–Rugby series.

It is important to reflect that in the equation

$$2d = (b - n) + (a - m) \qquad \text{(A)} \ddagger$$

the quantity d represents the discrepancies in time determined at two observatories and referred to the same meridian. Daily values of $2d$ are obtained from an intercomparison of signals, which intercomparison makes possible the elimination of the time of propagation of the radio waves as has been explained in the earlier paper. The only assumption that is made concerning the propagation of the electric waves in such an elimination is that the waves are propagated over essentially the same path and with the same wave velocity in both the west-to-east and east-to-west direction in the transatlantic transmission. A similar assumption as to velocity and

* " Apparent Lunar Effect in Time Determinations at Greenwich and Washington," *M.N.*, **93**, 444, 1933.

† " Variation Effect in Latitude Correlatable with the Moon," *Nature*, **131**, 437, 1933.

‡ "Apparent Lunar Effect in Time Determinations at Greenwich and Washington," *M.N.*, **93**, 444, 1933.

direction is, of course, made in the intercomparison of signals between France and England.

In the present discussion the time-signal data selected for analysis have been restricted to those days during the interval 1929–32 in which all three stations GBR (Rugby), FYL (Bordeaux) and NSS (Annapolis) were able to interchange signals under satisfactory conditions. As an arbitrary criterion for such a selection it was decided to exclude all data for those days on which $(y-x)-z > 0^{s}\cdot04$, where $x=$(GBR–NSS), $y=$(FYL–NSS) and $z=$(FYL–GBR).

The quantities x, y and z represent the three values of $2d$ in the equation (A) resulting from the three pairs of stations involved in the intercomparisons. The quantity a is the Washington correction to NSS (Annapolis), and b the corresponding Washington correction to the foreign station. These have been taken from the *U.S. Naval Observatory Time Signal Bulletin.* The quantities m and n designate respectively the correction to NSS as received abroad, and the correction to the signal sent from the European station. These have been taken for England from the *Admiralty Notices*, and for France from the *Bulletin Horaire.* In the comparison between Rugby and Annapolis it should be noted that it is the 17^{h} G.C.T. signal from Washington that is compared with the 18^{h} G.C.T. signal at Greenwich. In the case of the American–French comparison the 8^{h} G.C.T. signal at Washington is compared with the 8^{h} G.C.T. signal from Paris. In the English–French comparison the Greenwich signal at 18^{h} G.C.T. is compared with the French signal at 20^{h} G.C.T. The time of comparison would appear to have little to do with the phenomenon in question, provided that the interval between the exchanges is sufficiently short so that the clocks can safely carry the interval without appreciable error.

In the following table (Table I) there are printed the values of $2d$ as above defined derived from the intercomparison of the three stations Rugby, Annapolis and Bordeaux, for each date on which an exchange of signals was effected for all three stations. The fifth column gives the value $y-x$, resulting from taking the differences between the second and first columns. This, as already explained, gives the calculated $2d$ for FYL–GBR, and a direct comparison of this calculated value $y-x$ with the observed value z results in the differences exhibited in column six marked "diff." The two final columns in the table give the calculated hour-angle and declination of the Moon for the Greenwich meridian at Greenwich Civil Time (G.C.T.) 0^{h} for the date in question.

TABLE I

Date	x GBR–NSS	y FYL–NSS	z FYL–GBR	$(y-x)$	Diff.	Moon H.A.	Moon Dec.
1929						°	°
Feb. 2	18	20	−01	2	3	267	−16
5	14	14	−01	0	1	233	−25
6	12	14	−02	2	4	221	−26
18	19	12	−09	−7	2	78	24
19	19	11	−09	−8	1	63	26
21	13	6	−09	−7	2	33	25

Date	x GBR–NSS	y FYL–NSS	z FYL–GBR	$(y-x)$	Diff.	Moon H.A.	Moon Dec.
1929						°	°
Mar. 7	5	4	01	− 1	2	227	−26
12	14	13	00	− 1	1	168	− 4
13	16	15	00	− 1	1	157	2
18	9	15	07	6	1	93	26
19	5	13	02	8	6		
23	2	15	05	13	8		
Apr. 8	6	8	01	2	1	200	− 6
10	10	15	02	5	3	177	6
17	10	20	03	10	7		
18	7	11	02	4	2	66	21
25	2	7	04	5	1	348	−16
May 10	1	9	10	8	2	170	21
21	− 7	12	17	19	2	29	−10
31	− 9	17	22	26	4	273	−16
June 3	− 8	19	15	27	12		
10	− 4	12	13	16	3	142	26
July 10	0	15	15	15	0	133	16
22	−19	− 2	13	17	4	355	−25
23	−18	2	14	20	6		
30	− 2	5	08	7	1	268	15
Aug. 10	5	8	03	3	0	118	− 6
13	0	2	01	2	1	86	−21
16	− 2	6	04	8	4	50	−27
18	− 3	5	04	8	4		
26	− 9	3	07	12	5		
27	− 7	3	08	10	2	285	19
30	− 1	10	07	11	4	208	24
31	− 3	9	08	12	4	210	24
Sept. 11	1	11	11	10	1	93	−25
27	− 8	8	13	16	3	257	27
28	− 6	8	11	14	3	243	25
30	0	13	11	13	2	215	17
Oct. 1	− 1	14	13	15	2	204	11
3	5	21	13	16	3	181	− 2
4	3	17	14	14	0	170	− 8
5	− 1	15	14	16	2	159	−13
7	1	17	20	16	4	136	−22
9	1	18	17	17	0	111	−27
11	− 6	13	16	19	3	87	−27
19	− 3	1	08	4	4	357	11
21	2	8	07	6	1	332	21
25	5	14	06	9	3	273	26
Nov. 7	− 4	− 7	−05	− 3	2	118	−27
9	− 3	− 5	−02	− 2	0	94	−23
10	− 4	− 4	−02	0	2	83	−19

Date	x GBR–NSS	y FYL–NSS	z FYL–GBR	$(y-x)$	Diff.	Moon H.A.	Moon Dec.
1929						°	°
Nov. 11	− 3	− 5	−04	− 2	2	72	−15
14	5	1	−06	− 4	2	40	2
15	10	5	00	− 5	5		
17	12	7	−09	− 5	4	4	20
18	17	8	−09	− 9	0	350	24
20	13	6	−10	− 7	3	319	28
23	12	14	−07	2	9		
27	3	6	01	3	2	230	− 4
28	4	5	00	1	1	219	−10
30	3	4	01	1	0	197	−20
Dec. 1	− 2	1	03	3	0	185	−23
2	− 4	4	03	8	5		
3	6	9	01	3	2	161	−27
5	− 1	7	03	8	5		
13	− 7	5	05	12	7		
15	− 2	6	09	8	1	24	22
17	− 1	9	05	10	5		
1930							
Jan. 6	5	13	04	8	4	112	− 7
8	6	4	01	− 2	3	92	4
10	2	8	01	6	5		
12	7	10	04	3	1	43	24
13	11	13	02	2	0	28	27
15	16	9	−04	− 7	3	355	26
16	9	14	−02	5	7		
18	16	10	00	− 6	6		
19	11	12	−02	1	3	301	5
21	5	1	−05	− 4	1	278	− 7
23	4	2	−02	− 2	0	256	−18
Feb. 3	3	11	05	8	3	131	− 3
5	0	4	02	4	2	111	9
8	−12	− 5	09	7	2	75	23
10	− 7	6	12	13	1	46	28
11	1	10	07	9	2	30	27
12	2	10	10	8	2	15	24
16	3	− 3	10	− 6	16		
18	1	15	13	14	1	299	−11
19	− 4	11	13	15	2	287	−17
24	0	11	09	11	2	227	−27
25	3	10	07	7	0	214	−26
27	4	10	06	6	0	192	−19
Mar. 2	9	9	03	0	3	161	− 4
3	12	9	−01	− 3	2	151	2
4	9	11	01	2	1	140	8
5	4	7	−01	3	4	130	13
6	4	3	−02	− 1	1	118	18
7	3	4	00	1	1	106	23
12	10	2	−06	− 8	2	32	22
16	3	5	−02	2	4	342	− 3

Date	x GBR–NSS	y FYL–NSS	z FYL–GBR	$(y-x)$	Diff.	Moon H.A.	Moon Dec.
1930						°	°
Mar. 19	8	2	−03	− 6	3	307	−20
22	3	0	−01	− 3	2	270	−28
23	0	0	−01	0	1	257	−28
24	0	0	−01	0	1	245	−26
25	0	− 3	−04	− 3	1	233	−24
30	− 3	3	−04	6	10		
Apr. 5	1	− 6	−06	− 7	1	107	27
7	3	0	−08	− 3	5		
8	11	0	−06	−11	5		
9	5	− 2	−02	− 7	5		
10	− 2	− 2	01	0	1	38	13
12	6	8	00	2	2	14	0
13	6	9	00	3	3	2	− 6
17	2	− 1	−02	− 3	1	314	−26
19	2	− 4	02	− 2	4	288	−28
21	− 7	− 3	02	4	2	263	−25
23	− 4	− 2	03	2	1	241	−18
25	− 7	1	02	8	6		
May 1	− 4	4	09	8	1	153	25
7	−10	− 1	07	9	2	68	15
8	−10	− 4	06	6	0	56	9
10	−10	− 5	06	5	1	33	− 4
11	−13	− 8	04	5	1	22	−11
12	−15	−10	04	5	1	10	−16
14	− 9	− 2	05	7	2	346	−25
20	− 1	− 4	03	− 3	6		
21	− 4	− 5	−02	− 1	1	261	−15
23	− 7	− 7	−01	0	1	241	− 4
24	− 8	− 3	−04	5	9		
25	− 7	− 8	−05	− 1	4	221	8
26	−10	− 6	−02	4	6		
28	− 6	− 6	−02	0	2	185	23
29	− 3	− 3	−02	0	2	170	26
31	− 7	− 9	−02	− 2	0	139	27
June 1	− 8	−27	−02	−19	17		
2	− 5	− 8	−03	− 3	0	110	21
6	− 4	− 7	−03	− 3	0	62	− 3
11	− 5	1	−01	6	7		
12	− 2	1	00	3	3	350	−28
21	− 6	− 2	01	4	3	252	6
July 6	− 4	− 2	02	2	0	58	−19
15	− 7	− 3	05	4	1	311	−12
25	− 8	− 2	06	6	0	190	27
31	−12	− 2	10	10	0	111	− 7
Aug. 1	−12	− 2	08	10	2	100	−12
3	−13	− 4	06	9	3	76	−22
5	−10	3	08	13	5		
12	−10	2	16	12	4	331	− 8

Date	x GBR–NSS	y FYL–NSS	z FYL–GBR	$(y-x)$	Diff.	Moon H.A.	Moon Dec.
1930						°	°
Aug. 15	−10	− 2	05	8	3	302	8
18	−10	− 3	05	7	2	268	23
19	−11	0	07	11	4	254	26
20	−11	0	07	11	4	239	28
Sept. no dates.							
Oct. 1	− 5	2	02	7	5		
15	−10	4	11	14	3	272	27
17	−10	5	10	15	5		
20	−16	3	17	19	2	207	2
Nov. 12	− 8	− 1	01	7	6		
15	− 9	− 5	04	4	0	249	10
18	8	− 5	02	−13	15		
25	6	14	05	8	3	124	−26
28	5	10	04	5	1	91	−14
30	0	8	08	8	0	71	− 4
Dec. 2	6	16	09	10	1	51	8
4	− 1	14	14	15	1	29	19
5	− 5	12	15	17	2	17	23
6	− 6	7	11	13	2	3	26
8	− 1	21	17	22	5		
10	− 6	19	23	25	2	304	22
12	− 2	15	17	17	0	279	12
16	4	26	14	22	8		
19	4	19	17	15	2	195	−26
21	0	18	19	18	1	168	−28
24	2	20	15	18	3	132	−20
30	3	19	14	16	2	72	11
31	0	16	14	16	2	61	17
1931							
Jan. 1	1	16	14	15	1	49	21
2	0	14	9	14	5		
5	8	15	2	7	5		
6	11	17	3	6	3	336	24
7	11	16	5	5	0	322	19
8	4	18	7	14	7		
9	4	17	9	13	4	297	7
10	4	13	6	9	3	285	0
12	− 2	12	8	14	6		
13	− 3	10	4	13	9		
14	− 2	9	8	11	3	250	−18
15	4	10	6	6	0	225	−26
18	5	11	2	6	4	186	−27
19	5	10	0	5	5		
21	5	6	1	1	0	151	−17
26	2	7	4	5	1	102	10
27	1	6	2	5	3	92	15
28	0	8	3	8	5		
31	0	5	2	5	3	41	28

Date	x GBR–NSS	y FYL–NSS	z FYL–GBR	$(y-x)$	Diff.	Moon H.A.	Moon Dec.
1931						°	°
Feb. 4	− 6	1	− 1	7	8		
6	− 6	− 4	0	2	2	316	2
11	− 7	− 2	− 2	5	7		
24	− 5	0	1	5	4	111	19
25	− 7	− 1	3	6	3	100	23
Mar. 3	− 7	0	5	7	2	15	19
5	− 3	3	6	6	0	349	5
10	− 4	1	3	5	2	285	−25
14	− 7	− 1	3	6	3	233	−26
19	−10	− 4	2	6	4	180	− 4
21	− 6	− 5	2	1	1	108	12
26	− 8	2	2	10	8		
Apr. 1	1	8	− 4	7	11		
6	0	2	− 1	2	3	318	−23
7	0	9	− 6	9	15		
8	− 6	− 3	− 2	3	5		
13	− 6	− 9	− 2	− 3	1	229	−16
14	− 8	− 5	0	3	3	219	−11
20	0	1	2	1	1	159	21
21	− 4	0	− 2	4	6		
24	1	5	4	4	0	106	28
28	0	7	5	7	2	53	11
29	− 4	5	8	9	1	41	4
May 5	0	5	8	5	3	323	−28
6	− 3	10	9	13	4	309	−28
7	− 7	5	7	12	5		
8	− 6	1	4	7	3	282	−25
9	− 1	4	8	5	3	270	−22
15	− 6	4	11	10	1	211	10
16	− 6	6	11	12	1	200	15
19	− 5	8	12	13	1	164	27
22	− 2	11	10	13	3	122	26
23	− 2	6	9	8	1	109	23
26	−12	2	8	14	6		
29	− 7	3	8	10	2	36	−14
30	− 1	7	3	8	5		
31	− 1	3	4	4	0	10	−24
June 3	− 5	2	7	7	0	327	−28
4	2	6	7	4	3	314	−26
15	0	16	10	16	6		
24	−11	7	14	18	4	78	− 6
29	− 9	7	13	16	3	13	−28
July 6	−12	8	12	20	8		
8	− 9	5	12	14	2	271	6
9	− 6	5	12	11	1	261	12
10	− 5	6	12	11	1	251	17
11	− 9	7	10	16	6		
24	− 7	2	9	9	0	71	−22

Date	x GBR–NSS	y FYL–NSS	z FYL–GBR	$(y-x)$	Diff.	Moon H.A.	Moon Dec.
1931							
July 25	− 9	3	8	12	4	58°	−26°
30	− 4	1	6	5	1	352	−22
Aug. 7	− 1	5	2	6	4	271	20
10	0	0	1	0	1	233	28
11	1	0	− 1	− 1	0	218	28
12	− 2	0	0	2	2	204	26
15	− 3	− 4	− 1	− 1	0	163	11
17	1	− 1	− 1	− 2	1	138	− 3
24	2	4	− 3	2	5		
25	0	3	− 2	3	5		
30	− 3	− 4	1	− 1	2	340	− 2
Sept. 1	− 2	− 4	− 2	− 2	0	321	9
5	0	− 2	− 2	− 2	0	278	26
7	3	− 2	− 4	− 5	1	251	29
8	2	− 1	− 5	− 3	2	237	27
9	6	1	− 5	− 5	0	222	24
10	1	− 3	− 7	− 4	3	208	20
12	2	− 4	− 8	− 6	2	182	7
14	5	− 1	− 7	− 6	1	158	− 7
17	− 1	− 6	− 7	− 5	2	119	−24
20	3	− 7	− 9	−10	1	77	−28
21	1	− 9	− 9	−10	1	64	−27
24	2	− 6	−10	− 8	2	29	−15
25	2	− 5	− 8	− 7	1	19	− 9
28	0	− 5	− 6	− 5	1	351	7
29	− 4	− 3	− 3	1	4	341	13
Oct. 2	− 1	0	0	1	1	308	26
5	− 2	5	0	7	7		
13	− 2	− 3	0	− 1	1	165	−17
15	−10	− 3	2	7	5		
16	− 8	− 3	2	5	3	122	−28
19	1	3	− 1	2	3	81	−24
20	1	1	− 3	0	3	70	−21
21	0	1	− 2	1	3	59	−16
23	4	2	− 3	− 2	1	39	− 5
24	4	− 1	− 4	− 5	1	30	0
25	5	2	− 5	− 3	2	20	6
26	4	0	− 4	− 4	0	11	12
28	7	0	− 7	− 7	0	349	21
29	4	− 3	− 7	− 7	0	338	25
30	4	0	− 8	− 4	4	325	27
Nov. 1	4	0	− 7	− 4	3	299	28
2	4	0	− 6	− 4	2	285	27
3	6	0	− 7	− 6	1	272	23
4	7	5	− 6	− 2	4	259	18
5	7	6	− 6	− 1	5		
6	7	0	− 8	− 7	1	235	6
7	5	2	− 9	− 3	6		
8	8	1	− 9	− 7	2	211	− 8

Date	x GBR–NSS	y FYL–NSS	z FYL–GBR	$(y-x)$	Diff.	Moon H.A.	Moon Dec.
1931						°	°
Nov. 9	10	1	− 8	− 9	1	198	−15
10	8	2	− 8	− 6	2	184	−20
11	7	4	− 7	− 3	4	170	−25
12	5	0	− 6	− 5	1	155	−27
13	6	0	− 8	− 6	2	140	−29
14	11	5	− 8	− 6	2	126	−28
16	10	3	− 8	− 7	1	100	−22
17	9	3	−10	− 6	4	89	−17
18	9	5	−11	− 4	7		
19	10	6	− 8	− 4	4	69	− 7
20	7	4	− 6	− 3	3	59	− 1
21	7	5	− 5	− 2	3	50	5
23	11	7	− 4	− 4	0	30	15
24	10	8	− 6	− 2	4	19	20
25	10	5	− 7	− 5	2	8	24
27	10	7	− 5	− 3	2	342	28
28	11	5	− 7	− 6	1	329	28
29	9	6	− 6	− 3	3	315	27
30	11	8	− 4	− 3	1	302	24
Dec. 1	13	9	− 2	− 4	2	289	20
2	9	14	− 2	5	7		
4	4	8	2	4	2	254	1
6	9	6	− 3	− 3	0	230	−12
7	8	4	− 2	− 4	2	217	−18
8	7	7	− 2	0	2	203	−23
11	5	4	− 3	− 1	2	159	−28
12	6	6	− 4	0	4	145	−26
14	5	6	− 6	1	5		
20	10	6	−10	− 4	6		
21	9	2	−10	− 7	3	50	19
22	9	1	−12	− 8	4	39	23
23	9	0	− 9	− 9	0	26	26
27	11	3	−11	− 8	3	332	25

Irrespective of a distortion of the observer's meridian due to any lunar effect on the vertical at any of the stations at which the star observations were made, it will appear obvious that a comparison of signals exchanged between Rugby and Annapolis with the signals exchanged between Bordeaux and Annapolis gives at once a calculated difference between the Greenwich and the Paris clocks for the day in question. This calculated difference may then be compared with the actual observed difference between Greenwich and Paris by an examination of the time-signals exchanged between Rugby and Bordeaux. This serves as an immediate check on the degree of precision which may be ascribed to the signals exchanged on a given date. It is to be noted that the values in the "diff." column sometimes show large discrepancies between the observed and the calculated values. All such discrepancies have been carefully examined with recourse to the original data published, and the signal records for a given day, as previously stated, have been discarded for all three stations if on examination it is found that

the discrepancies in the checks, as judged from the "diff." column, are greater than $0^s.04$. This arbitrary value is selected as one sufficiently great to allow for variations and uncertainties due to errors in the third decimal place, and all reasonable uncertainties in making and recording the chronograph records. From the three years' observations (1929, 1930 and 1931) 323 time-signals have been exchanged at all three stations out of a possible number of 1064 dates during the interval selected. Restricting the data to those days on which the clock differences between Washington and Greenwich, and between Washington and Paris, check within $0^s.04$ with the direct comparison between Greenwich and Paris, the total number of days on which time data are available are reduced from 323 to 260 days.

The values of $2d$ for these days are the basis for the plotting of three curves for each of the three possible intercomparisons. In plotting the graphs running means of five successive values of $2d$ and the lunar hour-angles were used as was done in the earlier paper. The full-line curves in fig. 1 show the values of $2d$ for the Rugby–Annapolis intercomparison and represent the change of $2d$ with the hour-angle of the Moon for three zones of lunar declination: (*a*) when the Moon is south of the meridian from 12° S. to 28° S.; (*b*) when the Moon is near the equator from 12° S. to 12° N.; and (*c*) when the Moon is north of the equator from 12° N. to 28° N.; the hour-angle being taken for the Greenwich meridian at G.C.T. 0^h. These curves, as would be expected, show reasonable agreement with the curves previously published * on the less restricted data.

The broken-line curves in fig. 1 represent results based on the comparison of signals between Bordeaux and Annapolis, indicating an apparent variation in longitude between Paris and Washington. It will be at once apparent that the broken-line curves show essentially the same form as the full-line curves that represent the apparent variation of longitude between Greenwich and Washington as the Moon changes its hour-angle and declination. It will be noted that the characteristic reversal in phase between declination south and declination north which was emphasized in the previously published paper is well confirmed in the Paris–Washington series. This characteristic, as pointed out in the earlier paper, is strong evidence of a tidal effect in the region of latitude 45° on the surface of the Earth.

It will now be of special interest to see what variations with lunar hour-angle occur between Greenwich and Paris. The results of the direct comparison of the time-signals between FYL (Bordeaux) and GBR (Rugby) are shown in the full-line curves depicted in fig. 2. The broken-line curves are the results of calculating the variations in longitude between Greenwich and Paris by subtracting the GBR–NSS differences from the FYL–NSS differences. The dotted-line curves represent the differences between these observed and computed values and are labelled $o-c$. The fact that the $o-c$ curves show little variation from a horizontal line is a graphical argument for a satisfactory check in the observations.

It becomes apparent from a study of these graphs that no such varia-

* "Apparent Lunar Effect in Time Determinations at Greenwich and Washington, *M.N.*, **93**, 444, 1933.

tion of 2*d* with the lunar hour-angle takes place in the longitude between Greenwich and Paris as appears to take place between Washington and either of the European stations. There seems to be strong evidence that something happens across the Atlantic that does not happen across the

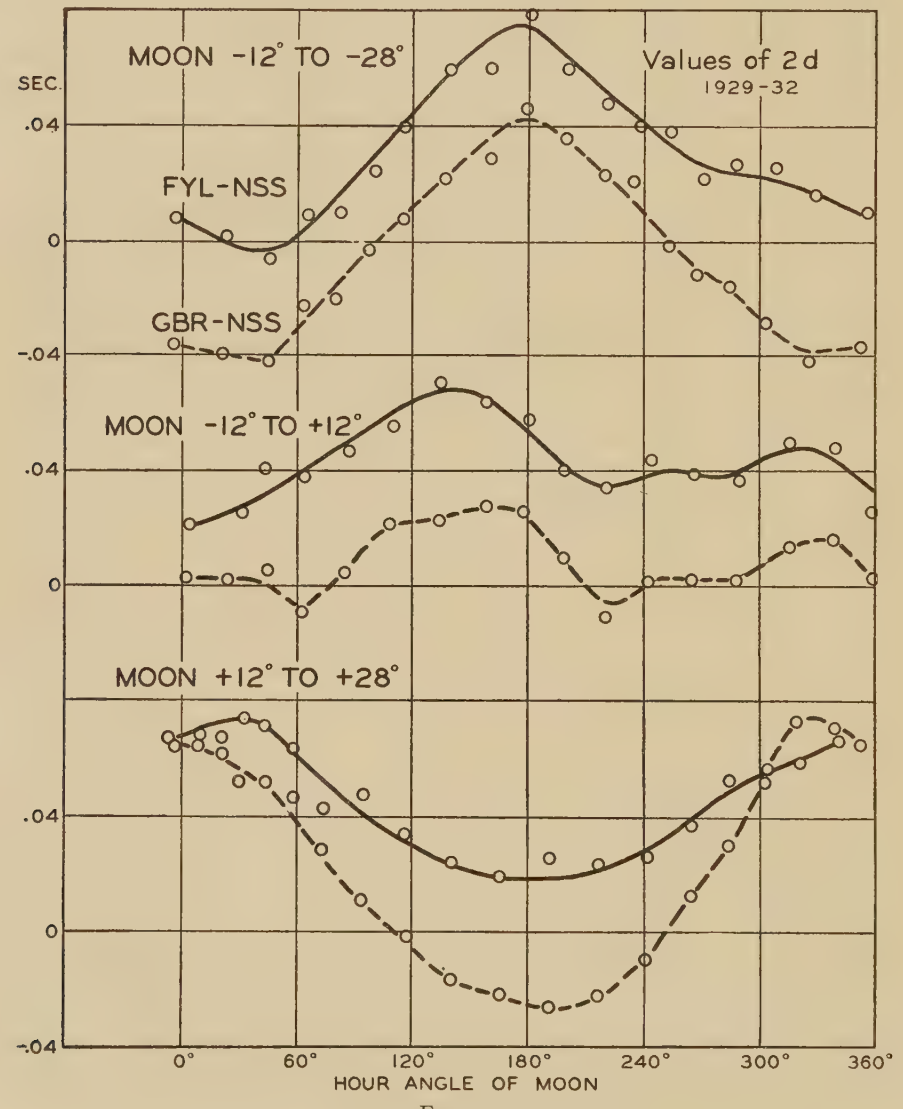

FIG. 1.

English Channel in introducing apparent changes in longitude with changes of the Moon's position. It may be too early to attempt an explanation, but it would appear that the cause could lie in small displacements of the observer's meridian, implying slight periodic changes in the geographical co-ordinates of the observatories involved, unless one were to suppose that the velocity of transmission of the radio waves in travelling from east to

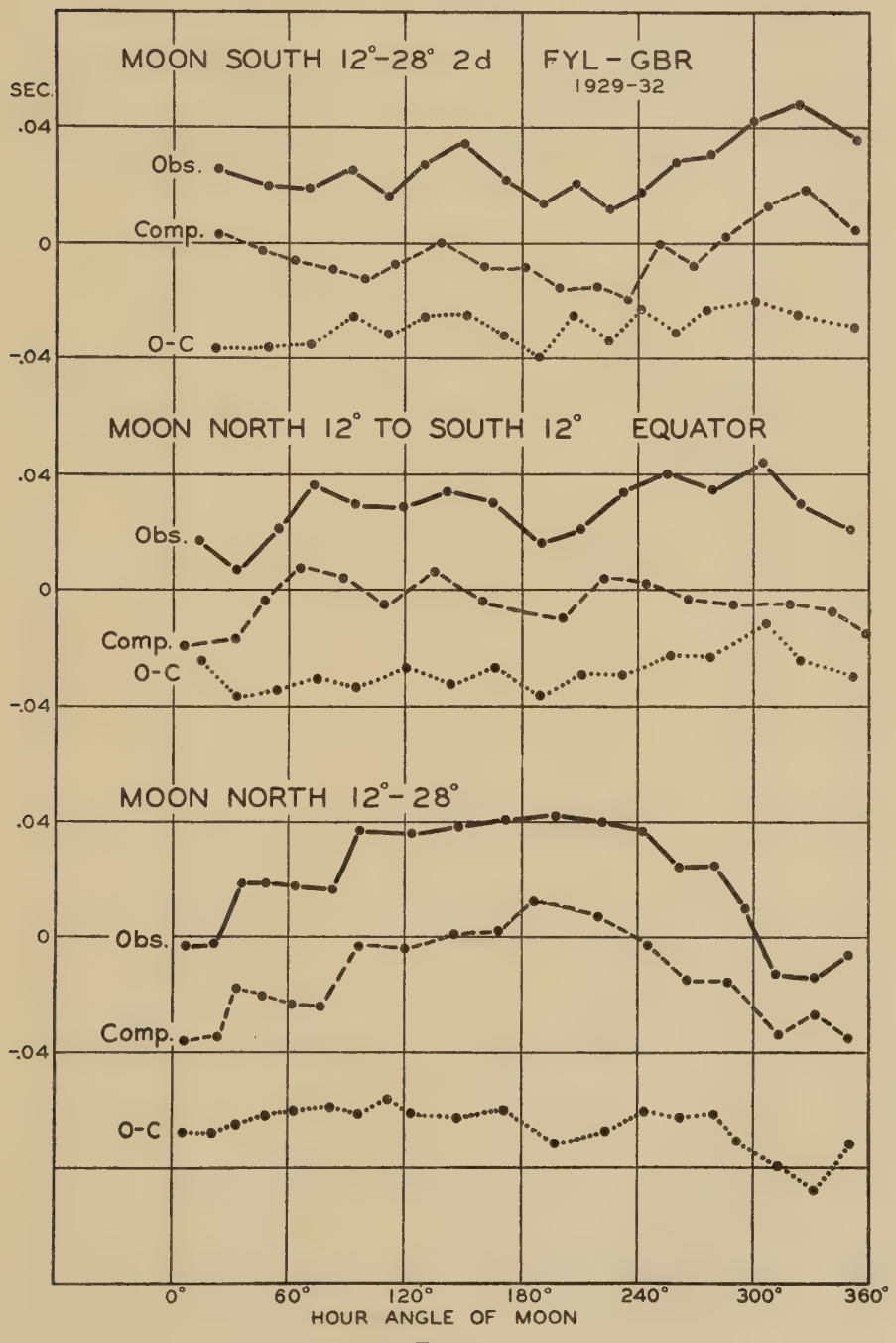
MOON SOUTH 12°-28° 2d
FYL - GBR
1929-32
SEC.
.04
0
-.04
Obs.
Comp.
O-C
MOON NORTH 12° TO SOUTH 12° EQUATOR
.04
0
-.04
Obs.
Comp.
O-C
MOON NORTH 12°-28°
.04
0
-.04
Obs.
Comp.
O-C
0°
60°
120°
180°
240°
300°
360°
HOUR ANGLE OF MOON

FIG. 2.

west is sensibly different from that of waves sent in the opposite direction or that the paths are notably different in the two directions.

Table II gives the values of the apparent lunar effect L which are derived from the curves of $2d$ for the different classes of declination, the lunar hour-angle being used as the argument. It has been suggested by E. W. Brown that if the apparent lunar effect here discussed is real, then an application of this L factor to the quantity $2d$, derived from each day's intercomparison of signals, should reduce the mean variation of $2d$. Such is found to be the case.

TABLE II

(Corrections to $2d$ derived from curves of the apparent lunar effect L)

	Dec. South −12° to −28°		Dec. near Equator −12° to +12°		Dec. North +12° to +28°	
	GBR–NSS L	FYL–NSS L	GBR–NSS L	FYL–NSS L	GBR–NSS L	FYL–NSS L
°						
0	−·035	·010	·001	·020	·065	·065
15	−·037	·003	·000	·024	·061	·072
30	−·039	−·003	·002	·028	·056	·072
45	−·040	−·005	·002	·033	·050	·069
60	−·032	·003	−·005	·039	·041	·063
75	−·020	·014	·000	·045	·027	·055
90	−·010	·025	·009	·052	·015	·049
105	·003	·037	·017	·060	·004	·040
120	·012	·048	·023	·065	−·005	·030
135	·021	·058	·027	·068	−·012	·025
150	·031	·067	·028	·067	−·017	·020
165	·040	·075	·026	·063	−·023	·018
180	·043	·074	·023	·055	−·025	·019
195	·038	·066	·013	·042	−·027	·020
210	·030	·057	·000	·040	−·024	·022
225	·020	·048	−·007	·037	−·018	·023
240	·010	·040	−·002	·040	−·010	·027
255	·000	·032	·003	·042	·000	·032
270	−·010	·025	·003	·040	·014	·040
285	−·020	·025	·002	·037	·027	·050
300	−·028	·025	·005	·042	·045	·055
315	−·034	·022	·013	·050	·060	·058
330	−·037	·016	·017	·050	·068	·063
345	−·037	·012	·015	·045	·067	·065
360	−·035	·010	·001	·020	·065	·065

As an example we give in Table III the values of $2d$ and of $2d - L$ derived from the satisfactory comparisons available during 1929, the first of the three years for which material has been used in this investigation. The table is divided into two parts. In the first part, based on the comparison GBR (Rugby)–NSS (Annapolis), it will be noted that systematic differences in the value of $2d$ appear to exist both before and after the application of the hypothetical lunar correction L. Disregarding the significance of these systematic differences, it is of interest to note that the summation of the

TABLE III

GBR–NSS

Date 1929	$2d$	$2d-L$	Date 1929	$2d$	$2d-L$	Date 1929	$2d$	$2d-L$
Feb. 2	·180	·190	Jul. 10	·000	·012	Oct. 9	·010	·003
5	·140	·125	22	−·190	−·155	11	−·060	−·050
6	·120	·100	30	−·020	−·034	19	−·030	−·031
18	·190	·163				21	·020	−·048
19	·190	·149	Aug. 10	·050	·027	25	·050	·036
21	·130	·074	13	·000	·010			
			16	−·020	·020	Nov. 7	−·040	−·052
Mar. 7	·050	·030	27	−·070	−·097	9	−·030	−·022
12	·140	·114	30	−·010	·014	10	−·040	−·025
13	·160	·133	31	−·030	−·006	11	−·030	−·010
						14	·050	·048
Apr. 8	·060	·050	Sept. 11	·010	·020	17	·120	·056
10	·100	·077	27	−·080	−·080	18	·170	·103
18	·070	·035	28	−·060	−·050	20	·130	·070
25	·020	·057	30	·000	·022	27	·030	·035
						28	·040	·045
May 10	·010	·033	Oct. 1	−·010	−·017	30	·030	−·008
21	−·070	−·072	3	·050	·027			
31	−·090	−·080	4	·030	·005	Dec. 1	−·020	−·063
			5	−·010	−·046	3	·060	·020
June 10	−·040	−·025	7	·010	−·011	15	−·020	−·078
						Sum	1·450	·843
						Mean	·027	·016

FYL–NSS

Feb. 2	·200	·175	Jul. 10	·150	·125	Oct. 9	·180	·138
5	·140	·096	22	−·020	−·030	11	·130	·105
6	·140	·092	30	·050	·010	19	·010	−·010
18	·120	·065				21	·080	·017
19	·110	·047	Aug. 10	·080	·015	25	·140	·100
21	·060	−·012	13	·020	−·005			
			16	·060	·065	Nov. 7	−·070	−·118
Mar. 7	·040	−·008	27	·030	−·020	9	−·050	−·020
12	·130	·067	30	·100	·078	10	−·040	−·060
13	·150	·085	31	·090	·068	11	−·050	−·064
						14	·010	−·021
Apr. 8	·080	·038	Sept. 11	·110	·085	17	·070	·004
10	·150	·095	27	·080	·048	18	·080	·015
18	·110	·051	28	·080	·053	20	·060	·002
25	·070	·058	30	·130	·108	27	·060	·022
						28	·050	·012
May 10	·090	·072	Oct. 1	·140	·099	30	·040	−·026
21	·120	·092	3	·210	·155			
31	·170	·145	4	·170	·110	Dec. 1	·010	−·064
			5	·150	·071	3	·090	·015
June 10	·120	·098	7	·170	·112	15	·060	−·012
						Sum	4·660	2·438
						Mean	·086	·045

values of $2d$ before applying the lunar correction is considerably greater than after the correction is applied. This is evidenced from the values obtained: $\Sigma(2d) = 1{\cdot}500$; $\Sigma(2d - L) = {\cdot}843$. The summation is based on 54 dates. The mean value of $2d$, before making any allowance for the lunar correction, is ·027; whereas after the lunar correction L has been applied the mean value is reduced to ·016. This appears indeed to indicate a significant improvement in the scattering of $2d$ with the application of the lunar correction L. In the second part of the table there are given similar values resulting from the series of comparisons FYL (Bordeaux)–NSS (Annapolis). In this case $\Sigma(2d) = 4{\cdot}660$ and $\Sigma(2d - L) = 2{\cdot}438$. The mean value of $2d$ for the FYL–NSS series is 0·086 before applying the lunar correction, and 0·045 after the lunar correction has been applied. The improvement here is even more marked than in the GBR–NSS series.

Inquiring into the systematic changes in the values of $2d$, an examination of the tables shows a significant seasonal variation in $2d$ which is particularly conspicuous in the Greenwich–Washington comparison. As so many factors are involved in the interpretation of any variation in the period commensurate with the year a discussion of this is postponed until a later paper. It is obvious that there are other sources of variation and apparent longitude differences than can be accounted for on the hypothesis of a lunar effect alone. It appears sufficient for the present to indicate that a reasonably substantial support for the correlation of small differences in longitude with the hour-angle and declination of the Moon, to which the earlier paper called attention, appears to exist in the French–American time comparisons over the same period.

Without considering at the moment geophysical or astronomical causes which could enter into such variations in longitude as here presented, it should be stressed that careful consideration has been given to any possible differences in lag introduced in receiving the signals of the foreign and the nearby stations. The order of magnitude of such lags as have been mentioned appears altogether too small to explain the variations here presented. Furthermore, if such values for variation in the lag of recording should exist, it would appear highly improbable that the values should show any correlation whatever with the Moon's hour-angle and declination.

For assistance in carrying on investigations of variations in latitude and longitude, acknowledgment is made of a grant from the Gould Fund of the National Academy of Sciences and aid provided through the Harvard Geophysical Committee.

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1935 *January.*

THE BRIGHTNESS OF PROMINENCES AS SHOWN BY THE SPECTROHELIOSCOPE.

George E. Hale.

At the meeting of the Royal Astronomical Society on 1934 December 14, as reported in the January number of *The Observatory*, Mr. Newbegin described his recently completed spectrohelioscope. As I have not yet had an opportunity to observe with this instrument, I cannot compare the brightness of prominences at the Sun's limb with those seen with the many spectrohelioscopes I have tested here. But a word on this subject may not be out of place.

As the image of a prominence is composed of a rapid succession of images of the narrow $H\alpha$ line produced by a pair of Anderson rotating square prisms, its apparent brightness will depend upon :

(1) The intrinsic brightness of the prominence itself, which of course varies greatly, from the faintest quiescent prominences to the most brilliant eruptive ones.
(2) The effective aperture of the spectrohelioscope.
(3) The coefficient of reflection of the cœlostat and spectrohelioscope mirrors.
(4) The brightness of the grating in the $H\alpha$ region of the first order spectrum.
(5) The widths of the first and second slits.
(6) The size of the Anderson prisms, which determines the breadth of the field of view.
(7) The radial motion of any portion of a prominence, which may throw the $H\alpha$ line partially or completely off from the second slit. This is readily corrected by the aid of the line-shifter, which is constantly in use by an observer seeking to detect active phenomena and to secure maximum contrast.
(8) The intensity of the diffuse light entering the observer's eye from the sky, the cœlostat mirrors and telescope lens, the collimating mirror (which should be completely concealed by a narrow screen), the grating, and the mirror which forms the image of $H\alpha$ on the second slit.

It goes without saying that a tube enclosing the solar beam and extending from the outer wall of the observing room to the first slit should protect the eye from reflections of the solar image by the slit or adjacent parts and that the observing room should be as dark as possible. The apparent brightness can also be increased by the use of a V-shaped or iris diaphragm, permitting the first slit to be shortened and excluding most of the light of the Sun's

disc. A carefully selected piece of red glass may also be useful, though I do not find it necessary.

The above obvious precautions generally suffice in the case of a standard spectrohelioscope equipped with a fairly good grating. Thus I have completed to-day the final tests of such an instrument, destined for use at the National Observatory in Athens, which shows admirably the fine details of the bright and dark flocculi on the Sun's disc and (with somewhat wider slits) the prominences at the limb.

With a given grating and magnifying power, one way to increase materially the apparent brightness of the monochromatic solar image is to decrease the breadth of the field of view. With the Anderson rotating prisms, this could be done by decreasing the width of the prisms. Indeed, an easily adjustable arrangement could be devised, permitting any one of three pairs of prisms, mounted on two shafts telescoping into one another, to be brought into place before the upper and lower slits.

On my own spectrohelioscope, however, I have retained a single pair of oscillating slits. A simple device permits the amplitude to be instantly changed through any desired range. By reducing the amplitude while observing I can greatly increase the brightness of a prominence or of any portion of the Sun's disc, thus permitting details otherwise invisible to be detected. The slits also have some advantages over prisms, in that they may be more easily made, adjusted, and cleaned without danger of scratching.

I take this opportunity to call attention to the most interesting phenomena revealed by the spectrohelioscope, some of which are illustrated in a previous paper.* These are the rapidly changing comet-like flocculi which may form in a few seconds in the neighbourhood of active sunspots, rise with considerable velocity and pursue curved paths, at changing speeds, until their black heads impinge on the outer boundary line of the penumbra. In some cases their heads seem to reach the umbra, but the frequency with which they terminate at the apparent edge of the penumbra is remarkable. Because of their radial velocity, the line-shifter must be constantly in use to reveal them throughout their length.

Pasadena :
1935 *February* 15.

* Hale, "The Spectrohelioscope and its Work. Part II. The Motions of the Hydrogen Flocculi near Sunspots," *Ap. J.*, **71**, 55, 1930.

ON THE IONISATION IN THE NEBULAR ENVELOPE SURROUNDING A STAR.

V. A. Ambarzumian.

(Communicated by E. A. Milne)

At present there is scarcely any doubt that the emission lines in the spectra of Be stars, as well as the Wolf-Rayet emission bands, have their origin in extensive and tenuous atmospheres which form some sort of nebular envelope surrounding these stars. Recently some observational data about the emission lines and bands and their intensities have been collected by several observers. The correct theoretical interpretation of these data will be possible only in terms of the ionisation theory for such extended atmospheres. No such theory exists at present. An attempt to give the first outline of such a theory is given in the present paper.

It is clear that the ionisation depends chiefly upon the density of the corresponding "ultra-violet" radiation at the given point of the atmosphere. As soon as we have estimated this density our problem is approximately solved. But the calculation of this density is possible only on the basis of the theory of radiative equilibrium in these atmospheres. Thus the problem of ionisation is essentially a problem of the theory of radiative equilibrium.

A theory of radiative equilibrium of a planetary nebula was proposed by the present writer in two previous papers.* However, there is an important difference between the types of the radiative equilibrium in planetaries and in the nebular envelope of relatively small radius surrounding the stars with emission lines.

In a few words this difference may be explained in the following way. In the previous papers we have shown that the transformation of the quanta of the ultra-violet continuous spectrum into the quanta of the resonance line of hydrogen atom (L_α) on the one hand, and the large optical thickness of planetaries in the resonance frequency on the other, lead to a very high value of the density of L_α-radiation in the inner parts of a planetary nebula. Nevertheless, though exceeding 10^9 times the density of the direct L_α-radiation from the star falling on the inner boundary of nebulæ, this density is still many thousand times smaller than the density of L_α-radiation at the surface of the central star, since the dilution factor W is of the order of 10^{-13}. Therefore the relative number of atoms in the second level is also small compared with that on the surface of the star, and we may still neglect the transitions from the second level to the higher levels.

We know indeed that the number of such transitions per second is proportional to $n_2\rho_{23}$, where n_2 is the number of atoms per c.c. in the second

* *M.N.*, **93**, 50, 1932 ; *Bull. de l'observatoire central à Poulkovo*, **13**, 3, 1933.

level and ρ_{23} is the density of the corresponding radiation. For n_2 we have approximately $n_2 \cong n_1 \frac{\rho_{12}}{\sigma_{12}}$, where ρ_{12} is the density of radiation corresponding to transition $1 \to 2$ and $\sigma_{12} = \frac{8\pi h\nu_{12}^3}{c^3}$. Therefore the number of transitions $2 \to 3$ is proportional to $n_1\bar{\rho}_{12}\rho_{23}$, while the number of transitions $1 \to 3$ is proportional to $n_1\rho_{13}$. The coefficient of proportionality is of the same order of magnitude in both cases.

Now on the surface of the star we have approximately

$$(\bar{\rho}_{13})_{\text{surf}} = (\bar{\rho}_{12})_{\text{surf}}(\bar{\rho}_{23})_{\text{surf}}. \tag{A}$$

In the nebula we have $(\bar{\rho}_{13})_{\text{neb}} \cong W(\bar{\rho}_{13})_{\text{surf}}$, $(\bar{\rho}_{23})_{\text{neb}} = W(\rho_{23})_{\text{surf}}$ and $(\bar{\rho}_{12})_{\text{neb}} = \epsilon(\bar{\rho}_{12})_{\text{surf}}$, where ϵ is a small quantity of the order of 10^{-4} as was mentioned above.

Therefore taking into account (A) we find

$$(\bar{\rho}_{12})_{\text{neb}}(\bar{\rho}_{23})_{\text{neb}} \cong \epsilon(\bar{\rho}_{13})_{\text{neb}},$$

or

$$n_1(\bar{\rho}_{12})_{\text{neb}}(\rho_{23})_{\text{neb}} \cong n_1 \frac{\nu_{23}^3}{\nu_{13}^3}\epsilon(\rho_{13})_{\text{neb}},$$

and we see that the number of the transitions $2 \to 3$ is a negligible fraction of the number of the transitions $1 \to 3$. At the same time a considerable portion of atoms arriving at the third state passes spontaneously to the second state, and we may say, therefore, that the number of transitions $2 \to 3$ is negligible compared with the number of transitions $3 \to 2$. In other words, we may neglect the number of the cyclic transitions of the type $1 \to 2 \to 3 \to 1$ compared with the number of the cyclic transitions of the type $1 \to 3 \to 2 \to 1$.

In accordance with this, the method developed in our previous papers where the transitions $1 \to 2 \to 3 \to 1$ were neglected, is applicable to the planetaries.

It cannot be applied, however, to the cases where the dilution of radiation is not so high as in planetaries. Thus it is in no case applicable to the gaseous envelopes of the Wolf-Rayet stars and Be stars. In the Wolf-Rayet stars the envelopes are immediately attached to the surfaces of the stars and the maximum of density is at the surface itself. In this case, for example, W is of the order of unity, and our method is certainly not applicable. At the same time we cannot restrict ourselves in this case to the consideration of monochromatic radiative equilibrium, as we do in the case of the reversing layers of stars for the resonance lines, making use of the fact that the cyclic transitions are relatively rare when compared with the transitions of the type $1 \to 2 \to 1$. In fact, though this is still true in the case of Wolf-Rayet stars, we cannot neglect the cyclic transitions, since these transitions only are responsible for the appearance of the emission lines (or bands).

At first sight it seems questionable whether it is necessary to take into

account the partial compensation of the "direct" cyclic processes, corresponding to the fluorescence phenomenon ($1 \rightarrow 3 \rightarrow 2 \rightarrow 1$), by reverse processes ($1 \rightarrow 2 \rightarrow 3 \rightarrow 1$), when we are convinced that our theory would have only a qualitative character owing to many other approximations of a physical and geometrical nature. It seems to us that for the question of the final intensities of the emission lines, it is of little difference whether we take into account the reverse processes or not. But if our purpose is the calculation of the ionisation in the envelopes, the difference between these two cases is very large. The exact formulæ we shall give below, but even now we may see that the degree of ionisation, which is proportional to the density of corresponding radiation, in the case of the complete compensation of the processes of both kinds can be represented according to Schuster's theory in the form :

$$\frac{n^+}{n} n_e = C \frac{t + 0{,}5}{t_1 + 1}, \tag{B}$$

where t is the optical depth of the given point of the envelope and t_1 the whole optical thickness, while in the case when each absorbed "ultra-violet" quantum is split up into quanta of small frequencies and no reverse process occurs, we shall have

$$\frac{n^+}{n} n_e = C' e^{-(\tau_1 - \tau)}. \tag{C}$$

In the case when the optical thickness of the envelope is large compared with unity, the formulæ (B) and (C) give us the ionisation of quite a different order of magnitude. It is possible that the exact formula will give something between (B) and (C). But the difference between (B) and (C) is too large to give us any possibility even of approximate estimation of the quantity under consideration.

In the present paper we are performing some calculations which, perhaps, may serve this purpose.

The Conditions of the Steady State.—We shall consider an atom in which the electron has only three energy levels $\epsilon_1 < \epsilon_2 < \epsilon_3$. Each of these levels has the corresponding weight g_1, g_2, g_3. If the density of matter is so low that we may neglect the super-elastic collisions, and each electronic transition is accompanied by the corresponding radiation of a light quantum, the conditions of the steady state may be written in the form :

$$\left.\begin{aligned} n_1\{B_{12}\rho_{12} + B_{13}\rho_{13}\} &= n_2 \frac{g_1}{g_2} B_{12}(\sigma_{12} + \rho_{12}) + n_3 \frac{g_1}{g_3} B_{13}(\sigma_{13} + \rho_{13}) \\ n_1 B_{13}\rho_{13} + n_2 B_{23}\rho_{23} &= n_3 \left\{ \frac{g_1}{g_3} B_{13}(\sigma_{13} + \rho_{13}) + \frac{g_2}{g_3} B_{23}(\sigma_{23} + \rho_{23}) \right\} \end{aligned}\right\}, \tag{1}$$

where n_k is the number of atoms in kth state in c.c., ρ_{ik} is the density of radiation with frequency $\nu_{ik} = \frac{|\epsilon_i - \epsilon_k|}{h}$, B_{ik} is Einstein's probability coefficients of transition from lower state i to the higher state k.

The quantities σ_{ik} are connected with ν_{ik} and the universal-constants by means of relation

$$\sigma_{ik} = \frac{8\pi h \nu_{ik}^3}{c^3}. \tag{2}$$

The equations (1) replace the equation of the radiative equilibrium in the case when the conditions of thermodynamical equilibrium are not fulfilled.

If we suppose now that the third energy level is not a discrete one, but corresponds to the case when the atom is ionised and the electron is in a free state, we shall introduce instead of the Einstein probability coefficients $B_{1\to 3}$ and $B_{2\to 3}$ from one discrete state to another, some probability coefficients of the photo-electric transitions defined in an appropriate way. Thus the whole number of the ionisations of atoms per c.c. during an interval of time dt is equal to

$$(n_1 b_{1\to 3}\rho_{13} + n_2 b_{2\to 3}\rho_{23})dt, \tag{3}$$

where ρ_{13} is the specific density of radiation for minimal frequency ν_{13}, which is necessary for the ionisation of the normal atom, and ρ_{23} is the specific density for minimum frequency ν_{23}, which is necessary for the ionisation of an excited atom. Strictly speaking, the number of ionisations, from the first level for example, depends not only on the specific density in the frequency ν_{13}, but also on the specific densities for all frequencies which satisfy the inequality $\nu > \nu_{13}$. However, we may suppose that the *relative* distribution of energy beyond the frequency ν_{13} is determined by a single parameter T ("temperature"), and assume that the expression (3) is valid under the condition that $b_{1\to 3}$ and $b_{2\to 3}$ are dependent on this parameter.

The number of the spontaneous recombinations during the same time dt is equal to the expression

$$n_3(a_{3\to 1} + a_{3\to 2})n_e dt, \tag{4}$$

where n_3 is now the number of ionised atoms, and $n_3 a_{3\to 1}dt$ (or $n_3 a_{3\to 2}dt$) is the probability of such recombination of a free electron with an ionised atom, which gives an atom in the first level (or in the second level). The simple considerations connected with the process of ionisation show that between our a–s and b–s there are the following relations :—

$$a_{3\to 1} = \frac{g_1\sigma_{13}}{g^+ G} b_{1\to 3}\,; \qquad a_{3\to 2} = \frac{g_2\sigma_{23}}{g^+ G} b_{2\to 3}, \tag{5}$$

where g^+ is the weight of the normal state of the ionised atom and

$$G = \frac{(2\pi m \kappa T)^{3/2}}{h^3}. \tag{6}$$

The expressions (3) and (4) for the number of transitions and the relations (5) and (6) make it possible for us to write in our case the conditions of the steady state in the form :

$$\left.\begin{aligned} n_1\{B_{12}\rho_{12}+b_{13}\rho_{13}\} &= n_2\frac{g_1}{g_2}B_{12}(\sigma_{12}+\rho_{12})+\frac{g}{g^+G}b_{13}(\sigma_{13}+\rho_{13})n_3n_e \\ n_1b_{13}\rho_{13}+n_2b_{23}\rho_{23} &= \frac{n_3n_e}{G}\left\{\frac{g_1}{g^+}b_{13}(\sigma_{13}+\rho_{13})+\frac{g_2}{g^+}b_{23}(\sigma_{23}+\rho_{23})\right\} \end{aligned}\right\}. \quad (7)$$

The equations (7) may be brought to the form (1) if we denote in (7)

$$g_3=\frac{g^+G}{n_e} \quad (8)$$

and write $B_{1\to 3}$ and B_{23} instead of b_{13} and b_{23}. The only difference is that the coefficients B_{13} and B_{23} as well as g_3 depend in our case on the parameter T. Therefore we can use the equations (1) as the starting-point of our calculations.

In the course of our discussions we shall suppose that B_{13}, B_{23} and g_3 are constant, since the range of the relative variation of these quantities is undoubtedly very small compared with the range of variation of other quantities entering into our investigation. This circumstance will have the consequence that we may obtain from our theory the information, not about the *distribution* of the "ultra-violet" radiation in the spectrum, but only about the approximate mean value of its intensity.

The Approximate Form of the Solution of the Equations of the Steady State.—Each specific density ρ_{ik} we may write in the form $\rho_{ik}=\bar{\rho}_{ik}\cdot\sigma_{ik}$. The quantities $\bar{\rho}_{ik}$ are dimensionless. According to Planck's law, at the surface of stars we shall have

$$\bar{\rho}_{ik}=\frac{1}{e^{\frac{h\nu_{ik}}{kT}}-1}.$$

Let us suppose that $\nu_{12}>2kT$ and $\nu_{23}>2kT$.* Then the quantities $\bar{\rho}_{ik}$ are even at the surface of the star small compared with unity. We have approximately at the surface $\bar{\rho}_{13}=\bar{\rho}_{12}\bar{\rho}_{23}$, and therefore if we call $\bar{\rho}_{12}$ and $\bar{\rho}_{23}$ small quantities of the first order, $\bar{\rho}_{13}$ will be of the second order. In the envelope $\bar{\rho}_{ik}$ may scarcely attain values many times exceeding their surface values. Therefore we may always regard them as small quantities, and especially $\bar{\rho}_{13}$ as a small quantity of the second order.

The equations of the transfer of radiation for the frequencies ν_{12} and ν_{13}, which we shall consider in the following paragraph, contain the following two expressions :—

$$\frac{\frac{g_1}{g_2}n_2}{n_1-\frac{g_1}{g_2}n_2} \quad \text{and} \quad \frac{\frac{g_1}{g_3}n_3}{n_1-\frac{g_1}{g_3}n_3}.$$

These ratios are to be calculated from the equations (1). The corresponding expressions are somewhat complicated. But they attain extraordinary simplicity if we restrict ourselves to the members of the first and

* These conditions are fulfilled almost in every case, which is of practical interest.

of the second order and neglect the higher terms. The calculations are too long to be reproduced here. The results are

$$\frac{\frac{g_1}{g_2}n_2}{n_1-\frac{g_1}{g_2}n_2}=\bar{\rho}_{12}+\gamma(\bar{\rho}_{13}-\bar{\rho}_{12}\bar{\rho}_{23}), \tag{9}$$

$$\frac{\frac{g_1}{g_3}n_3}{n_1-\frac{g_1}{g_3}n_3}=\bar{\rho}_{13}+\beta(\bar{\rho}_{12}\bar{\rho}_{23}-\bar{\rho}_{13}), \tag{10}$$

where the constants γ and β have the following values :—

$$\gamma=\frac{g_2B_{13}B_{23}\sigma_{13}\sigma_{23}}{B_{12}\sigma_{12}[g_1B_{13}\sigma_{13}+g_2B_{23}\sigma_{23}]}, \tag{11}$$

$$\beta=\frac{g_2B_{23}\sigma_{23}}{g_1B_{13}\sigma_{13}+g_2B_{23}\sigma_{23}}=\frac{B_{12}\sigma_{12}}{B_{13}\sigma_{13}}\gamma. \tag{12}$$

The approximation which we have applied has a very simple physical meaning. We may consider indeed the atomic transitions of the type $1\to2\to1$ as simple processes of the scattering of the quanta with the frequency ν_{12} by normal atoms. The more complicated processes of the type $1\to2\to3\to1$ or $1\to2\to3\to2\to1$ we may consider as the processes of the collision of the normal atom with two quanta $h\nu_{12}$ and $h\nu_{23}$. After the collision of such type we may have either one quantum with the frequency ν_{13}, or again two quanta $h\nu_{12}$ and $h\nu_{23}$. There may also occur the collisions of three or more quanta with a normal atom. Such are, for example, the processes of the type $1\to2\to3\to2\to3\to1$, where two quanta $h\nu_{23}$ and a quantum $h\nu_{12}$ collide with the normal atom. If, however, the density of radiation for every frequency is small we may neglect all collisions where more than two quanta take part. Assuming further that the density of radiation of the frequency ν_{13} is small even when compared with ρ_{12} and ρ_{23}, we may neglect also such collisions with two quanta at once, when at last one of two quanta has the frequency ν_{13}. Thus we shall obtain again the equations (9) and (10).

From this point of view the "classical" theory of monochromatic radiative equilibrium, which, as it was emphasised by some writers, is applicable only to the resonance lines, is the first approximation where only the small quantities of the first order are taken into account. This approximation is sufficient to explain on general lines the formation of absorption lines, but cannot give the explanation of the existence of the emission lines. We know (according to Rosseland) that such an explanation is impossible unless three frequencies and the cyclic transitions are taken into account. The theory outlined in the present paper is a second approximation, since the terms of the second order are taken into account. It has general character, while the theory given in the previous works of the writer is applicable only to the special cases when, owing to very strong dilution of radiation,

we may neglect the term $\bar{\rho}_{12}\bar{\rho}_{23}$ compared with $\bar{\rho}_{13}$. The present theory, though still very far from accurate, will give in every case the main features of the phenomena of emission-lines.

The Equations of Transfer.—For each frequency appearing in our consideration we may write the corresponding equation of transfer. But it is superfluous to do it for the frequency ν_{23}, since we may suppose that the total optical thickness of the gaseous envelope in this frequency is certainly small compared with unity. In fact this frequency corresponds to the ionisation from an excited state, and therefore the optical thickness in it will be of the same order of magnitude as the optical thickness arising from the *general opacity* of the envelope, since this opacity is caused chiefly by such bound-free transitions from excited states. The optical thickness arising from the general opacity reaches the values of the order of unity only in the photosphere of the star. The same will be true for the frequency ν_{23}, and the envelope will be transparent for this frequency. Therefore we may write directly

$$\bar{\rho}_{23} = \frac{W}{e^{\frac{h\nu_{23}}{\kappa T}} - 1}, \tag{13}$$

where W is the coefficient of dilution.

The equations of the transfer of radiation for the frequencies ν_{12} and ν_{13} as it is known * may be written in the form :

$$\frac{dI_{12}}{\frac{h\nu_{12}}{\Delta\nu_{12}}\left(n_1 - \frac{g_1}{g_2}n_2\right)\frac{B_{12}}{c}ds} = -I_{12} + \frac{g_1}{g_2}\frac{c\sigma_{12}}{4\pi}\frac{n_2}{n_1 - \frac{g_1}{g_2}n_2}, \tag{14}$$

$$\frac{dI_{13}}{\frac{h\nu_{13}}{\Delta\nu_{13}}\left(n_1 - \frac{g_1}{g_3}n_3\right)\frac{B_{13}}{c}ds} = -I_{13} + \frac{g_1}{g_3}\frac{c\sigma_{13}}{4\pi}\frac{n_3}{n_1 - \frac{g_1}{g_3}n_3}, \tag{15}$$

where I_{12} and I_{13} are the specific intensities of radiation with frequencies ν_{12} and ν_{13}, ds is the element of the trajectory of the ray, $\Delta\nu_{12}$ is the width of the resonance line and $\Delta\nu_{13}$ the effective width of the corresponding absorption line.

If we introduce the absorption coefficient α per unit of volume for the frequency ν_{12},

$$\alpha = \frac{h\nu_{12}}{\Delta\nu_{12}}\left(n_1 - \frac{q_1}{q_2}n_2\right)\frac{B_{12}}{c}, \tag{16}$$

we obtain

$$\frac{dI_{12}}{\alpha ds} = -I_{12} + \frac{g_1}{g_2}\frac{c\sigma_{12}}{4\pi}\frac{n_2}{n_1 - \frac{g_1}{g_2}n_2}, \tag{17}$$

$$\frac{dI_{13}}{\alpha ds} = \frac{\nu_{13}\Delta\nu_{12}}{\nu_{12}\Delta\nu_{13}}\frac{n_1 - \frac{g_1}{g_3}n_3}{n_1 - \frac{g_1}{g_2}n_2}\frac{B_{13}}{B_{12}}\left\{-I_{13} + \frac{g_1}{g_3}\frac{c\sigma_{13}}{4\pi}\frac{n_3}{n_1 - \frac{g_1}{g_3}n_3}\right\}. \tag{18}$$

* Milne, *M.N.*, **88**, 493, 1928.

It is convenient to introduce instead of intensities I_{12} and I_{13} the dimensionless quantities ϕ and ψ:

$$I_{12}=\frac{c\sigma_{12}}{4\pi}\psi; \qquad I_{13}=\frac{c\sigma_{13}}{4\pi}\psi. \tag{19}$$

Then the equations (17) and (18) obtain the simple form:

$$\frac{d\phi}{\alpha ds}=-\phi+\frac{\frac{g_1}{g_2}n_2}{n_1-\frac{g_1}{g_2}n_2}, \tag{20}$$

$$\frac{d\psi}{\alpha ds}=\frac{\nu_{13}\Delta\nu_{12}}{\nu_{12}\Delta\nu_{13}}\frac{n_1-\frac{g_1}{g_3}n_3}{n_1-\frac{g_1}{g_2}n_2}\frac{B_{13}}{B_{12}}\left\{-\psi+\frac{\frac{g_1}{g_3}n_3}{n_1-\frac{g_1}{g_3}n_3}\right\}. \tag{21}$$

It is clear that between ϕ and ψ from one side and $\bar{\rho}_{12}$ and $\bar{\rho}_{13}$ from another we have the following relations :—

$$\bar{\rho}_{12}=\frac{1}{4\pi}\int\phi d\omega\,; \qquad \bar{\rho}_{13}=\frac{1}{4\pi}\int\psi d\omega, \tag{22}$$

where $d\omega$ is the element of the solid angle and the integration is carried over all directions.

If $\bar{\rho}_{13}$ is a small quantity of the second order, the quantity ψ will be also of the second order. The quantity in brackets in (21) is then also of the second order, and in the factor before these brackets we may freely neglect the small quantities of the first order and write

$$\frac{n_1-\frac{g_1}{g_3}n_3}{n_1-\frac{g_1}{g_2}n_2}=1.$$

If we denote further,

$$\frac{\nu_{13}\Delta\nu_{12}}{\nu_{12}\Delta\nu_{13}}\frac{B_{13}}{B_{12}}=q,$$

we may write, taking into account (9), (10) and (22) instead of (20) and (21),

$$\frac{d\phi}{\alpha ds}=-\phi+\frac{1}{4\pi}\int[\phi+\gamma(\psi-\bar{\rho}_{23}\phi)]d\omega, \tag{23}$$

$$\frac{d\psi}{q\alpha ds}=-\psi+\frac{1}{4\pi}\int[\psi+\beta(\bar{\rho}_{23}\phi-\psi)]d\omega. \tag{24}$$

It remains now to solve these equations, where ϕ and ψ are the functions of co-ordinates and of the direction of radiation. However, the solution may be actually performed only in the case when the boundary conditions for ϕ and ψ and the function α of co-ordinates are given.

The Geometrical Model.—The equations (23) and (24) have general character and are applicable to all atmospheric problems with any (constant or variable) coefficient of dilution. However, we exclude from further considerations the Wolf-Rayet stars owing to the fact that the high velocity of ejection of matter from them leads to an inequality of frequencies of the given line in different parts of the atmosphere. Therefore the problem in this case cannot be reduced to a one-dimensional one.

In the calculations given below we shall restrict ourselves again to the cases when the dilution factor is small compared with unity. Let us suppose, for example, that $W < \frac{1}{100}$. In this case we may with safety use the method of the reduction of the spherical problem to a plane problem developed by Professor Milne.* The only difference will be in the circumstance that we shall count the diffuse radiation and the direct radiation coming from the star together, while in Milne's work, as well as in the writer's papers, they were treated separately. This will make the method slightly less accurate. But the calculations are in this case not so complicated.

Let us introduce the optical depth at the distance r from the centre of the star :

$$\tau = \int_r^{r_2} \alpha dr,$$

where r_2 is the outer boundary of the nebular shell. If further r_1 is the distance of the inner boundary from the centre, we shall write

$$\tau_1 = \int_{r_1}^{r_2} \alpha dr.$$

τ_1 is the whole optical thickness of the shell.

Using the approximation of the Schwarzschild-Schuster type and introducing the average values ϕ and ϕ' of ϕ for the outward and inward directions of radiation and the corresponding average values ψ and ψ' of the quantity ψ, we have instead of (23) and (24) the following approximate equations :—

$$\frac{1}{2}\frac{d\phi}{d\tau} = \phi - \tfrac{1}{2}\{(\phi+\phi') + \gamma[\psi+\psi' - \bar{\rho}_{23}(\phi+\phi')]\}, \tag{25}$$

$$\frac{1}{2}\frac{d\phi'}{d\tau} = \tfrac{1}{2}\{(\phi+\phi') + \gamma[\psi+\psi' - \bar{\rho}_{23}(\phi+\phi')]\} - \phi', \tag{26}$$

$$\frac{1}{2q}\frac{d\psi}{d\tau} = \psi - \tfrac{1}{2}\{(\psi-\psi') + \beta[\bar{\rho}_{23}(\phi+\phi') - (\psi+\psi')]\}, \tag{27}$$

$$\frac{1}{2q}\frac{d\psi'}{d\tau} = \tfrac{1}{2}\{(\psi+\psi') + \beta[\bar{\rho}_{23}(\phi+\phi') - (\psi+\psi')]\} - \psi'. \tag{28}$$

At the outer boundary we have the following conditions :—

$$\phi'(0) = \psi'(0) = 0. \tag{D}$$

* E. A. Milne, *Zs. f. Astrophysik*, **1**, 98, 1930.

At the inner boundary we have for the resonance frequency the condition that the resulting flux vanishes,

$$\phi(\tau_1) - \phi'(\tau_1) = 0, \tag{E}$$

since we may neglect the direct radiation from the star in this frequency. For the "ultra-violet" radiation we have at this boundary

$$\psi(\tau_1) - \psi'(\tau_1) = \psi_0, \tag{F}$$

owing to the fact that $\psi(\tau_1)$ contains the direct radiation from the star. The constant ψ_0 is connected with the amount πS of the ultra-violet energy falling on each square centimetre of the inner surface by means of the relation:

$$\psi_0 = \tfrac{1}{2}\frac{4\pi S}{c\sigma_{13}} = 2W(\bar{\rho}_{13})_{\text{surf}}.$$

It remains now to solve the equations (25), (26), (27) and (28), taking into account the boundary conditions (D), (E) and (F).

The Solution of the Equations.—Adding (25) with (26) and (27) with (28) we obtain

$$\frac{1}{2}\frac{d(\phi+\phi')}{d\tau} = \phi - \phi', \tag{29}$$

$$\frac{1}{2q}\frac{d(\psi+\psi')}{d\tau} = \psi - \psi'. \tag{30}$$

Subtracting (26) from (25) and (28) from (27) we have

$$\frac{1}{2}\frac{d(\phi-\phi')}{d\tau} = -\gamma[\psi+\psi' - \bar{\rho}_{23}(\phi+\phi')], \tag{31}$$

$$\frac{1}{2q}\frac{d(\psi-\psi')}{d\tau} = -\beta[\bar{\rho}_{23}(\phi-\phi') - (\psi+\psi')]. \tag{32}$$

Differentiating (29) and (30) and comparing with (31) and (32) we obtain the following system of two second-order equations for $\phi+\phi'$ and $\psi+\psi'$:—

$$\frac{1}{4}\frac{d^2(\phi+\phi')}{d\tau^2} = -\gamma[\psi+\psi' - \bar{\rho}_{23}(\phi+\phi')], \tag{33}$$

$$\frac{1}{4q^2}\frac{d^2(\psi+\psi')}{d\tau^2} = \beta[\psi+\psi' - \bar{\rho}_{23}(\phi+\phi')]. \tag{34}$$

In Milne's model the linear thickness of the nebular shell is assumed small compared with the distance from the centre. Therefore it is consistent with this model to put $W = \text{const}$, or according to (13), $\bar{\rho}_{23} = \text{const}$.

In this case the general solution of the system of equations (33) and (34) has the form:

$$\left.\begin{aligned} \phi+\phi' &= C_1 + C_2\tau + C_3 e^{\lambda\tau} + C_4 e^{-\lambda\tau} \\ \psi+\psi' &= \bar{\rho}_{23}(C_1 + C_2\tau) - \frac{q^2\beta}{\gamma}(C_3 e^{\lambda\tau} + C_4 e^{-\lambda\tau}) \end{aligned}\right\}, \tag{35}$$

where

$$\lambda = 2\sqrt{\gamma\bar{\rho}_{23} + \beta q^2}.$$

Using (29) and (30) we obtain now :

$$\left.\begin{aligned} \phi - \phi' &= \frac{C_2}{2} + \frac{C_3\lambda}{2}e^{\lambda\tau} - \frac{C_4\lambda}{2}e^{-\lambda\tau} \\ \psi - \psi' &= \frac{\bar{\rho}_{23}}{2q}C_2 - \frac{q\beta}{2\gamma}\lambda C_3 e^{\lambda\tau} + \frac{q\beta}{2\gamma}\lambda C_4 e^{-\lambda\tau} \end{aligned}\right\}. \tag{36}$$

Comparing (36) with (E) and (F) we have

$$C_2 + C_3\lambda e^{\lambda\tau_1} - C_4\lambda e^{-\lambda\tau_1} = 0, \tag{37}$$

$$\frac{\bar{\rho}_{23}}{2q}C_2 - \frac{q\beta}{2\gamma}\lambda C_3 e^{\lambda\tau_1} + \frac{q\beta}{2\gamma}\lambda C_4 e^{-\lambda\tau_1} = \psi_0. \tag{38}$$

From (35) and (36) we have further

$$2\phi' = \left(C_1 - \frac{C_2}{2}\right) + C_2\tau + C_3\left(1 - \frac{\lambda}{2}\right)e^{\lambda\tau} + C_4\left(1 + \frac{\lambda}{2}\right)e^{-\lambda\tau}, \tag{39}$$

$$2\psi' = \left(C_1 - \frac{C_2}{2q}\right)\bar{\rho}_{23} + C_2\bar{\rho}_{23}\tau - \frac{q^2\beta}{\gamma}\left[\left(1 - \frac{\lambda}{2q}\right)C_3 e^{\lambda\tau} + \left(1 + \frac{\lambda}{2q}\right)C_4 e^{-\lambda\tau}\right]. \tag{40}$$

Therefore the boundary conditions (D) may be written in the form :

$$\left(C_1 - \frac{C_2}{2}\right) + C_3\left(1 - \frac{\lambda}{2}\right) + C_4\left(1 + \frac{\lambda}{2}\right) = 0, \tag{41}$$

$$\left(C_1 - \frac{C_2}{2q}\right)\bar{\rho}_{23} - \frac{q^2\beta}{\gamma}\left[\left(1 - \frac{\lambda}{2q}\right)C_3 + \left(1 + \frac{\lambda}{2q}\right)C_4\right] = 0. \tag{42}$$

From the conditions (37), (38), (41) and (42) we can determine the coefficients C_1, C_2, C_3 and C_4. We have

$$\left.\begin{aligned} C_1 = {} & \frac{q\psi_0}{\bar{\rho}_{23}}\frac{1}{1 + \dfrac{q^2\beta}{\gamma\bar{\rho}_{23}}} \\ & - \left[\left(1 - \frac{\lambda}{2}\right)e^{-\lambda\tau_1} + \left(1 + \frac{\lambda}{2}\right)e^{\lambda\tau_1}\right]\frac{(q-1)\psi_0}{\bar{\rho}_{23}\left(1 + \dfrac{q^2\beta}{\gamma\bar{\rho}_{23}}\right)} \\ & - \left\{\left[\left(1 + \frac{\lambda}{2}\right) + \frac{q^2\beta}{\gamma\bar{\rho}_{23}}\left(1 + \frac{\lambda}{2q}\right)\right]\left(1 - \frac{\lambda}{2}\right) - \left[\left(1 - \frac{\lambda}{2}\right) + \frac{q^2\beta}{\gamma\bar{\rho}_{23}}\left(1 - \frac{\lambda}{2q}\right)\right]\left(1 + \frac{\lambda}{2}\right)\right\}\frac{2q\psi_0}{\gamma\bar{\rho}_{23}}\frac{1}{1 + \dfrac{q^2\beta}{\gamma\bar{\rho}_{23}}} \\ & \Big/ \left[\left(1 + \frac{\lambda}{2}\right) + \frac{q^2\beta}{\gamma\bar{\rho}_{23}}\left(1 + \frac{\lambda}{2q}\right)\right]e^{\lambda\tau_1} + \left[\left(1 - \frac{\lambda}{2}\right) + \frac{q^2\beta}{\gamma\bar{\rho}_{23}}\left(1 + \frac{\lambda}{2q}\right)\right]e^{-\lambda\tau_1} \end{aligned}\right\} \tag{43}$$

$$\left.\begin{aligned}
C_2 &= \frac{2q\psi_0}{\bar{\rho}_{23}} \frac{1}{1+\dfrac{q^2\beta}{\gamma\bar{\rho}_{23}}} \\
C_3 &= -\frac{\dfrac{(1-q)\psi_0}{\bar{\rho}_{23}\left(1+\dfrac{q^2\beta}{\gamma\bar{\rho}_{23}}\right)} e^{-\lambda\tau_1} + \left[\left(1+\frac{\lambda}{2}\right)+\frac{q^2\beta}{\gamma\bar{\rho}_{23}}\left(1+\frac{\lambda}{2q}\right)\right]\frac{2q\psi_0}{\lambda\bar{\rho}_{23}} \dfrac{1}{1+\dfrac{q^2\beta}{\gamma\bar{\rho}_{23}}}}{\left[\left(1+\frac{\lambda}{2}\right)+\frac{q^2\beta}{\gamma\bar{\rho}_{23}}\left(1+\frac{\lambda}{2q}\right)\right]e^{\lambda\tau_1}+\left[\left(1-\frac{\lambda}{2}\right)+\frac{q^2\beta}{\gamma\bar{\rho}_{23}}\left(1-\frac{\lambda}{2q}\right)\right]e^{-\lambda\tau_1}} \\
C_4 &= -\frac{\dfrac{(1-q)\psi_0}{\bar{\rho}_{23}\left(1+\dfrac{q^2\beta}{\gamma\bar{\rho}_{23}}\right)} e^{\lambda\tau_1} + \left[\left(1-\frac{\lambda}{2}\right)+\frac{q^2\beta}{\gamma\bar{\rho}_{23}}\left(1-\frac{\lambda}{2q}\right)\right]\frac{2q\psi_0}{\gamma\bar{\rho}_{23}} \dfrac{1}{1+\dfrac{q^2\beta}{\gamma\bar{\rho}_{23}}}}{\left[\left(1+\frac{\lambda}{2}\right)+\frac{q^2\beta}{\gamma\bar{\rho}_{23}}\left(1+\frac{\lambda}{2q}\right)\right]e^{\lambda\tau_1}+\left[\left(1-\frac{\lambda}{2}\right)+\frac{q^2\beta}{\gamma\bar{\rho}_{23}}\left(1-\frac{\lambda}{2q}\right)\right]e^{-\lambda\tau_1}}.
\end{aligned}\right\} \quad (43)$$

Ionisation in the Envelope.—Our purpose is to find the ionisation in the envelope. When the envelope is transparent to radiation of the frequency ν_{13}, the ionisation is to be calculated according to the simple formulæ given by Eddington and Rosseland. However, the most interesting case is that when the envelope is not transparent to this radiation, *i.e.* the optical thickness in the frequency ν_{13}, which is equal to $q\tau_1$, is large compared with unity. Thus $\tau_1 >> \frac{1}{q}$. Usually $\frac{1}{q}$ is of the order of 10^4. Therefore, if the dilution of radiation is not as high as in planetaries, the quantity $e^{-\lambda\tau_1}$ is exceedingly small and $e^{\lambda\tau}$ exceedingly large. This circumstance gives the possibility of simplifying the expressions for C_3 and C_4, neglecting small quantities. Thus we obtain from the formulæ given above the following approximate values :—

$$C_3 = \frac{2q\psi_0}{\lambda\bar{\rho}_{23}} \frac{1}{1+\dfrac{q^2\beta}{\gamma\bar{\rho}_{23}}} e^{-\lambda\tau_1}; \qquad C_4 = -\frac{\dfrac{(1-q)\psi_0}{\bar{\rho}_{23}\left(1+\dfrac{q^2\beta}{\gamma\bar{\rho}_{23}}\right)}}{\left(1+\frac{\lambda}{2}\right)+\frac{q^2\beta}{\gamma\bar{\rho}_{23}}\left(1+\frac{\lambda}{2q}\right)}. \quad (44)$$

It is clear now that the expression

$$C_3 e^{\lambda\tau} + C_4 e^{-\lambda\tau}$$

is always very small except near $\tau = 0$ and $\tau = \tau_1$. In all other parts of the envelope, practically speaking in the whole envelope, we may write, neglecting these terms, instead of (35):

$$\left.\begin{aligned}
\phi + \phi' &= C_1 + C_2\tau \\
\psi + \psi' &= \bar{\rho}_{23}(C_1 + C_2\tau)
\end{aligned}\right\}. \quad (45)$$

Let us fix our attention on the case when the dilution of radiation is not high and therefore $\bar{\rho}_{23} >> q^2$.* In this case we have simply

$$C_2 = \frac{2q\psi_0}{\bar{\rho}_{23}}, \tag{46}$$

since β and γ are approximately of the same order of magnitude.

Taking into account that λ is small compared with unity, we have also

$$C_1 \simeq \frac{\psi_0}{\bar{\rho}_{23}}. \tag{47}$$

Therefore

$$\left.\begin{aligned} \phi + \phi' &\simeq \frac{2\psi_0}{\bar{\rho}_{23}}(q\tau + \tfrac{1}{2}) \\ \psi + \psi' &\simeq 2\psi_0(q\tau + \tfrac{1}{2}) \end{aligned}\right\}. \tag{48}$$

We may now obtain the approximate values of $\bar{\rho}_{12}$ and $\bar{\rho}_{13}$. We have

$$\bar{\rho}_{12} = \tfrac{1}{2}(\phi + \phi'); \qquad \bar{\rho}_{23} = \tfrac{1}{2}(\psi + \psi'),$$

and $\tau = \dfrac{t}{q}$ where t is the optical depth in the frequency ν_{13}. Thus :

$$\left.\begin{aligned} \bar{\rho}_{12} &\simeq \frac{\psi_0}{\bar{\rho}_{23}}(t + \tfrac{1}{2}) \\ \bar{\rho}_{13} &\simeq \psi_0(t + \tfrac{1}{2}) \end{aligned}\right\}. \tag{49}$$

If we remember now that

$$\psi_0 = 2W(\bar{\rho}_{13})_{\text{surf}}; \qquad \bar{\rho}_{23} = W(\bar{\rho}_{23})_{\text{surf}}; \qquad (\bar{\rho}_{13})_{\text{surf}} = (\bar{\rho}_{12})_{\text{surf}}(\bar{\rho}_{23})_{\text{surf}},$$

we can write instead of (49)

$$\left.\begin{aligned} \bar{\rho}_{12} &\simeq 2(\bar{\rho}_{12})_{\text{surf}}(t + \tfrac{1}{2}) \\ \bar{\rho}_{13} &\simeq 2W(\bar{\rho}_{13})_{\text{surf}}(t + \tfrac{1}{2}) \end{aligned}\right\}. \tag{50}$$

We see now that if our assumptions are verified the density of radiation in the resonance frequency in the envelope is larger than the same density on the surface of the star. But further investigations are necessary to examine how far this important conclusion remains valid when the super-elastic collisions in the envelope are taken into consideration.

Introducing (50) in (10) we find the degree of ionisation :

$$\frac{\dfrac{g_1}{g_3}n_3}{n_1 - \dfrac{g_1}{g_3}n_3} \simeq \bar{\rho}_{13} \simeq W(\bar{\rho}_{13})_{\text{surf}} \cdot (2t + 1);$$

or approximately :

$$\frac{g_1}{g_3}\frac{n_3}{n_1} \simeq W(\bar{\rho}_{13})_{\text{surf}}(2t + 1). \tag{51}$$

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* We have $\bar{\rho}_{23} = W(\bar{\rho}_{23})_{\text{surf}}$. Further, $q \simeq 10^{-4}$. If $(\bar{\rho}_{23})_{\text{surf}}$ is of the order of 10^{-2}, the inequality in text is satisfied when $W >> 10^{-6}$. It is certainly not satisfied in the case of planetaries.

According to (8) and (6) :

$$g_3 = \frac{g^+}{n_e} \frac{(2\pi m\kappa T)^{3/2}}{h^3}.$$

Further, approximately :

$$(\bar{\rho}_{13})_{\text{surf}} = e^{-\frac{h\nu_{13}}{kT}}.$$

Introducing these expressions into (51) we obtain finally :

$$\frac{n_3}{n_1} n_e = \frac{g^+}{g_1}(2t+1) W \frac{(2\pi m\kappa T)^{3/2}}{h^3} e^{-\frac{h\nu_{13}}{\kappa T}}. \tag{52}$$

We conclude that the ionisation decreases very slowly in the outward direction.

Astronomical Observatory of the University, Leningrad :
1934 *December.*

FOURIER SERIES FOR THE PERIODIC ORBITS AROUND THE TRIANGULAR LIBRATION POINTS.

Peder Pedersen.

(Communicated by E. Strömgren)

1. *Introduction.*—It is well known that infinitesimal periodic orbits around the two triangular libration points in the restricted problem of three bodies exist for values of the mass-product not exceeding $\frac{1}{27}$, the sum of the two finite masses being taken as unit mass. For values of the two finite masses satisfying the condition just mentioned there will in general exist two kinds of periodic orbits, short-period orbits and long-period orbits. When the mass-product has exactly the limiting value $\frac{1}{27}$ the short-period and long-period orbits coincide, the period being $\sqrt{2}$ times the time of revolution of the two finite masses.

The periodic orbits referred to have been found by the method of solving the variational equations corresponding to the differential equations of motion. Investigations of this kind have been carried out by Charlier,* Plummer,† Martin ‡ and others.

The method just mentioned has the drawback that the resulting period is a function of the two masses only, being independent of the dimensions of the periodic orbit. Only by considering terms of higher order than the first is it possible to determine how the period changes with increasing size of the periodic orbits. It follows from a more detailed investigation that

* Charlier, *Die Mechanik des Himmels II*, 1907.
† Plummer, *M.N.*, **62**, 6–17, 1901 ; *M.N.*, **92**, 442–448, 1932.
‡ M. Martin, *A.N.*, **244**, 161–170, 1931.

it will be necessary to retain terms up to the third order in order to be able to discuss the problem of the variation of period with size.

Investigations of this kind have been carried out for special values of the finite masses by E. W. Brown,* H. R. Willard † and the author.‡ In a paper in 1911 E. W. Brown has treated the case that the value of the mass-product is close to $\frac{1}{27}$. Making use of Hill's transformation and retaining terms of the third order with regard to the orbit dimension, he shows that periodic orbits around the two triangular libration points exist for values of the mass-product greater than $\frac{1}{27}$ also, only no infinitesimal orbits exist. Further, it is shown that for values of the mass-product close to $\frac{1}{27}$ the period in the short-period orbits decreases with increasing size of the orbits, while the period in the long-period orbits increases. In the author's paper in 1933 the same special case is investigated with the aid of Fourier series. The results found agree with those reached by Brown. Willard's work in 1913 is a continuation of Brown's, the same method being used. The mass-ratio of the two finite masses has, however, been chosen so as to correspond to the mass-ratio Jupiter-Sun. This ratio being taken equal to 1 : 1048, the mass-product is about 1/1050. The short-period orbits only are considered, and it is shown that the period decreases as the size of the orbit increases. As all terms up to the third order are calculated, the shape of the periodic orbits is determined. With the aid of a diagram the development of the orbits is shown up to greater distances from the libration point.

Already in an earlier paper in 1911 E. W. Brown § had treated the long-period orbits corresponding to the short-period orbits considered by Willard. The method used allowed a determination of the shape of the periodic orbits at greater distances from the libration point, provided that the periodic orbit is not far from a circle with unit radius and its centre at the position of the greater mass. The last-mentioned condition is fulfilled for long-period orbits for small values of the mass-product, but not at all for short-period orbits. Later on B. Thüring ‖ in two papers treated the same problem. For the long-period orbits considered the period increases with increasing size of the orbits.

Investigations of a more general character concerning the existence of the periodic orbits around the triangular libration points have been carried out by Buck,¶ Greaves ** and Wintner.††

For values of the mass-product greater than $\frac{1}{27}$ infinitesimal periodic orbits around the triangular libration points do not exist, as was already mentioned. It is evident from the variational equations, however, that for these values of the mass-product asymptotic orbits exist, *i.e.* orbits that

* E. W. Brown, *M.N.*, **71**, 492–502, 1911.
† H. R. Willard, *M.N.*, **73**, 471–484, 1913.
‡ *M.N.*, **94**, 167–185, 1933.
§ E. W. Brown, *M.N.*, **71**, 438–454, 1911.
‖ B. Thüring, *A.N.*, **238**, 357–368, 1930 ; *A.N.*, **243**, 181–194, 1931.
¶ F. R. Moulton, *Periodic Orbits*, 299–324, 1920.
** W. M. H. Greaves, *M.N.*, **82**, 139–149, 1922.
†† A. Wintner, *Math. Zeitschrift*, **32**, 641–667, 1930 ; *Publikationer fra Københavns Observatorium*, No. 71.

converge towards the libration point through an infinite number of revolutions around it. Whether these asymptotic orbits at the same time are periodic is a question that cannot be solved with the aid of the variational equations, but only starting from the differential equations of motion. In a number of papers E. Strömgren * has investigated these asymptotic orbits with the aid of the method of numerical integration. For the special value of the mass-product that is used in the numerical investigations of the restricted problem of three bodies at the Copenhagen Observatory, viz. $\frac{1}{4}$, it has been possible to demonstrate the existence of orbits that are both asymptotic and periodic. These asymptotic-periodic orbits have then served as starting-points in finding whole groups of periodic orbits around the triangular libration points.

When the value of the mass-product passes the critical value $\frac{1}{27}$ the infinitesimal orbits are changed from periodic orbits into asymptotic orbits. A clear representation of this change has been given in another paper by E. Strömgren.†

The present paper will be concerned with the problem, in continuation of the author's paper in 1933, to calculate Fourier series for the infinitesimal periodic orbits around the triangular libration points, valid for all values of the mass-product, for which the periodic orbits exist, terms up to the third order with regard to the size of the orbit being retained. The problems investigated in the work of E. W. Brown and of H. R. Willard referred to, as well as in the author's earlier paper, are therefore special cases of the problem treated in this paper. In carrying out the complicated computations necessary it has been a great help that the procedure is exactly the same as in the author's former paper, so that it has been possible to obtain valuable checks by inserting into the relations obtained the special values $\omega^2 = \frac{1}{2}$ and $\gamma = k$, for which the problem of the present paper reduces to the problem of the former paper.

2. *Determination of the Terms of the First Order.*—In my previous paper the following two differential equations governing the motion of the infinitesimal mass in the neighbourhood of the libration point L_4 were derived

$$\begin{aligned}
&\ddot{\xi} - 2\dot{\eta} - \tfrac{3}{4}\xi - \tfrac{3}{4}\sqrt{3}(1-2\mu)\eta \\
&\quad - \tfrac{21}{16}(1-2\mu)\xi^2 + \tfrac{3}{8}\sqrt{3}\xi\eta + \tfrac{33}{16}(1-2\mu)\eta^2 \\
&\quad + \tfrac{378}{32}\xi^3 + \tfrac{75}{32}\sqrt{3}(1-2\mu)\xi^2\eta - \tfrac{123}{32}\xi\eta^2 - \tfrac{45}{32}\sqrt{3}(1-2\mu)\eta^3 = 0, \\
&\ddot{\eta} + 2\dot{\xi} - \tfrac{3}{4}\sqrt{3}(1-2\mu)\xi - \tfrac{9}{4}\eta \\
&\quad + \tfrac{3}{16}\sqrt{3}\xi^2 + \tfrac{33}{8}(1-2\mu)\xi\eta + \tfrac{9}{16}\sqrt{3}\eta^2 \\
&\quad + \tfrac{255}{32}\sqrt{3}(1-2\mu)\xi^3 - \tfrac{123}{32}\xi^2\eta - \tfrac{135}{32}\sqrt{3}(1-2\mu)\xi\eta^2 - \tfrac{3}{32}\eta^3 = 0.
\end{aligned} \tag{1}$$

Terms up to the third order are retained in the series expressions. The Jacobian integral corresponding to (1) is

* E. Strömgren, *Publikationer fra Köbenhavns Observatorium*, No. 47, 1924; No. 61, 1929; No. 64, 1929; No. 67, 1930.

† E. Strömgren, *Publikationer fra Köbenhavns Observatorium*, No. 70, 1930.

$$\dot{\xi}^2+\dot{\eta}^2=2+\tfrac{3}{4}\xi^2+\tfrac{3}{2}\sqrt{3}(1-2\mu)\xi\eta+\tfrac{9}{4}\eta^2$$
$$+\tfrac{7}{8}(1-2\mu)\xi^3-\tfrac{3}{8}\sqrt{3}\xi^2\eta-\tfrac{3\,3}{8}(1-2\mu)\xi\eta^2-\tfrac{3}{8}\sqrt{3}\eta^3 \qquad (2)$$
$$-\tfrac{3\,7}{6\,4}\xi^4-\tfrac{2\,5}{1\,6}\sqrt{3}(1-2\mu)\xi^3\eta+\tfrac{1\,2\,3}{3\,2}\xi^2\eta^2+\tfrac{4\,5}{1\,6}\sqrt{3}(1-2\mu)\xi\eta^3+\tfrac{3}{6\,4}\eta^4-C.$$

We now put

$$\left.\begin{aligned}\xi&=\overset{\text{II}}{a_0}+\overset{\text{I}}{a_1}\cos\omega t+\overset{\text{I}}{a_{-1}}\sin\omega t+\overset{\text{II}}{a_2}\cos 2\omega t+\overset{\text{II}}{a_{-2}}\sin 2\omega t\\&\qquad+\overset{\text{III}}{a_3}\cos 3\omega t+\overset{\text{III}}{a_{-3}}\sin 3\omega t,\\ \eta&=\overset{\text{II}}{b_0}+\overset{\text{I}}{b_1}\cos\omega t+\overset{\text{I}}{b_{-1}}\sin\omega t+\overset{\text{II}}{b_2}\cos 2\omega t+\overset{\text{II}}{b_{-2}}\sin 2\omega t\\&\qquad+\overset{\text{III}}{b_3}\cos 3\omega t+\overset{\text{III}}{b_{-3}}\sin 3\omega t.\end{aligned}\right\} \qquad (3)$$

The roman numbers indicate the order of the coefficients. The problem now is to determine seven coefficients a and seven coefficients b in such a way that (1) is satisfied by the series expressions (3).

Retaining terms of the first order only in (1) and (3) and introducing (3) in (1), we get the four equations of condition

$$\left.\begin{aligned}(\omega^2+\tfrac{3}{4})\,.\,a_1+\tfrac{3}{4}\sqrt{3}(1-2\mu)\,.\,b_1 &\qquad +2\omega\,.\,b_{-1}=0,\\ \tfrac{3}{4}\sqrt{3}(1-2\mu)\,.\,a_1+(\omega^2+\tfrac{9}{4})\,.\,b_1 &\qquad -2\omega\,.\,a_{-1}=0,\\ -2\omega\,.\,b_1+(\omega^2+\tfrac{3}{4})\,.\,a_{-1}+\tfrac{3}{4}\sqrt{3}(1-2\mu)\,.\,b_{-1} &=0,\\ 2\omega\,.\,a_1+\tfrac{3}{4}\sqrt{3}(1-2\mu)\,.\,a_{-1}+(\omega^2+\tfrac{9}{4})\,.\,b_{-1} &=0.\end{aligned}\right\} \qquad (4)$$

With the aid of the two first equations (4) we can express a_{-1} and b_{-1} in terms of a_1 and b_1. Introducing these expressions in the last two equations (4) we get

$$\left.\begin{aligned}[\omega^4-\omega^2+\tfrac{2\,7}{1\,6}-\tfrac{2\,7}{1\,6}(1-2\mu)^2]\,.\,a_1&=0,\\ [\omega^4-\omega^2+\tfrac{2\,7}{1\,6}-\tfrac{2\,7}{1\,6}(1-2\mu)^2]\,.\,b_1&=0.\end{aligned}\right\} \qquad (5)$$

From equations (5) it appears that a necessary condition for periodic solutions is

$$\omega^4-\omega^2+\tfrac{2\,7}{1\,6}-\tfrac{2\,7}{1\,6}(1-2\mu)^2=0. \qquad (6)$$

This equation yields real values of ω, provided

$$\mu(1-\mu)\leqq\tfrac{1}{2\,7}. \qquad (7)$$

If, however,

$$\mu(1-\mu)>\tfrac{1}{2\,7}, \qquad (8)$$

then equation (6) has no real root, and in this case no infinitesimal periodic orbit around L_4 will exist. However, as was shown in my previous paper, there will, for values of the mass-product that are only little greater than $\frac{1}{2\,7}$, exist groups of periodic orbits with interior limiting orbits of finite size enclosing the libration point L_4.

In this paper we shall deal exclusively with such values of μ that satisfy the condition (7). For such values of μ equation (6) will yield real values of ω. If these values of ω are introduced in (5), then equations (5) will

be satisfied for all values of a_1 and b_1, which quantities can therefore be considered as constants of integration. For a_1 and b_1 we choose the values

$$a_1 = \epsilon, \qquad b_1 = 0, \tag{9}$$

where ϵ is a small quantity of the first order. Introducing the values of a_1 and b_1 in (4) we can determine a_{-1} and b_{-1}. Introducing the expressions for the coefficients thus found in (3), we get

$$\left.\begin{aligned} \xi &= \epsilon \cos \omega t + \frac{3}{8\omega}\sqrt{3}(1-2\mu)\epsilon \sin \omega t, \\ \eta &= \frac{-4\omega^2 - 3}{8\omega}\epsilon \sin \omega t, \end{aligned}\right\} \tag{10}$$

representing an elliptic orbital motion.

3. *Determination of the Terms of the Second Order.*—We now turn to the determination of the terms of the second order in the Fourier expansions. For the sake of brevity we shall put

$$\gamma = 1 - 2\mu. \tag{11}$$

We can now write the equation of condition (6) in the form

$$27\gamma^2 = 16\omega^4 - 16\omega^2 + 27. \tag{12}$$

Making use of this equation we can in the calculations that follow substitute for γ^2, γ^3, γ^4 . . . expressions containing the first power of γ only.

The procedure employed in calculating the coefficients of the second order is identical with the procedure employed in my previous paper. The coefficient scheme there given will be used in the following calculations. With the aid of the expressions for a_1, b_1, a_{-1} and b_{-1} just found we can compute the quantities in columns 1, cos $2\omega t$ and sin $2\omega t$ of the coefficient scheme corresponding to ξ^2, $\xi\eta$ and η^2. Introducing these quantities in the differential equations (1) we get two equations for the determination of a_0 and b_0 and four equations for the determination of a_2, b_2, a_{-2} and b_{-2}. Solving these equations we get

$$\left.\begin{aligned} a_0 &= \frac{\gamma}{256\omega^2}[-132\omega^2 - 99]\epsilon^2, \\ b_0 &= \frac{\sqrt{3}}{768\omega^2}[32\omega^4 + 60\omega^2 + 27]\epsilon^2. \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} a_2 &= \frac{\gamma}{256\omega^2(5\omega^2-1)}[-76\omega^4 - 13\omega^2 + 9]\epsilon^2, \\ b_2 &= \frac{\sqrt{3}}{2304\omega^2(5\omega^2-1)}[736\omega^6 - 884\omega^4 + 823\omega^2 - 243]\epsilon^2, \\ a_{-2} &= \frac{\sqrt{3}}{288\omega(5\omega^2-1)}[20\omega^4 - 65\omega^2 - 36]\epsilon^2, \\ b_{-2} &= \frac{\gamma}{32\omega(5\omega^2-1)}[-33\omega^2 + 10]\epsilon^2. \end{aligned}\right\} \tag{14}$$

4. *Determination of the Terms of the Third Order.*—Using the values of a_0, b_0, a_2, b_2, a_{-2} and b_{-2} thus found together with the values of a_1, b_1, a_{-1} and b_{-1} we can now compute the quantities in the columns $\cos 3\omega t$ and $\sin 3\omega t$ of the coefficient scheme corresponding to ξ^2, $\xi\eta$, η^2, ξ^3, $\xi^2\eta$, $\xi\eta^2$ and η^3. Introducing these quantities in the differential equations (1), we get four equations for the determination of a_3, b_3, a_{-3} and b_{-3}. Solving these equations we get

$$\left.\begin{aligned}
a_3 &= \frac{1}{9216\omega^2(5\omega^2-1)(10\omega^2-1)}[4752\omega^8+5732\omega^6+27566\omega^4 \\
&\qquad -7621\omega^2-297]\epsilon^3, \\
b_3 &= \frac{\sqrt{3}\gamma}{1536\omega^2(5\omega^2-1)(10\omega^2-1)}[2748\omega^6-1915\omega^4-446\omega^2+99]\epsilon^3, \\
a_{-3} &= \frac{\sqrt{3}\gamma}{24576\omega^3(5\omega^2-1)(10\omega^2-1)}[-5744\omega^8-12900\omega^6+24768\omega^4 \\
&\qquad -3091\omega^2+135]\epsilon^3, \\
b_{-3} &= \frac{1}{73728\omega^3(5\omega^2-1)(10\omega^2-1)}[112064\omega^{10}-109344\omega^8+302700\omega^6 \\
&\qquad -12596\omega^4-25875\omega^2+3483]\epsilon^3.
\end{aligned}\right\} \tag{15}$$

5. *Determination of q_1, q_2, q_3 and q_4.*—We can further compute the quantities in the columns $\cos\omega t$ and $\sin\omega t$ of the coefficient scheme corresponding to ξ^2, $\xi\eta$, η^2, ξ^3, $\xi^2\eta$, $\xi\eta^2$ and η^3. Introducing these quantities in the differential equations (1) we get four equations for the more exact determination of a_1, b_1, a_{-1} and b_{-1}:

$$\left.\begin{aligned}
(\omega^2+\tfrac{3}{4})a_1 + \tfrac{3}{4}\sqrt{3}\gamma b_1 + 2\omega b_{-1} &= q_1\epsilon, \\
\tfrac{3}{4}\sqrt{3}\gamma a_1 + (\omega^2+\tfrac{9}{4})b_1 - 2\omega a_{-1} &= q_2\epsilon, \\
-2\omega b_1 + (\omega^2+\tfrac{3}{4})a_{-1} + \tfrac{3}{4}\sqrt{3}\gamma b_{-1} &= q_3\epsilon, \\
2\omega a_1 + \tfrac{3}{4}\sqrt{3}\gamma a_{-1} + (\omega^2+\tfrac{9}{4})b_{-1} &= q_4\epsilon,
\end{aligned}\right\} \tag{16}$$

where q_1, q_2, q_3 and q_4 are small quantities of the second order determined by the equations

$$\left.\begin{aligned}
4608\omega^2(5\omega^2-1)q_1 &= [13168\omega^8-4880\omega^6+21517\omega^4 \\
&\qquad +19146\omega^2-3969]\epsilon^2, \\
1536\omega^2(5\omega^2-1)q_2 &= [596\omega^6+4399\omega^4+2280\omega^2-513]\sqrt{3}\gamma\epsilon^2, \\
6144\omega^3(5\omega^2-1)q_3 &= [5392\omega^8-12132\omega^6-400\omega^4 \\
&\qquad +7179\omega^2-1215]\sqrt{3}\gamma\epsilon^2, \\
18432\omega^3(5\omega^2-1)q_4 &= [17600\omega^{10}+42336\omega^8+24796\omega^6 \\
&\qquad +82020\omega^4+45279\omega^2-10935]\epsilon^2.
\end{aligned}\right\} \tag{17}$$

6. *Determination of ω.*—From the two first equations (16) we can find a_{-1} and b_{-1} expressed in terms of a_1 and b_1. Introducing these expressions in the two last equations (16), we get

$$\left.\begin{aligned}
[\omega^4-\omega^2+\tfrac{27}{16}-\tfrac{27}{16}\gamma^2]a_1 &= [(\omega^2+\tfrac{9}{4})q_1-2\omega q_4-\tfrac{3}{4}\sqrt{3}\gamma q_2]\epsilon, \\
[\omega^4-\omega^2+\tfrac{27}{16}-\tfrac{27}{16}\gamma^2]b_1 &= [(\omega^2+\tfrac{3}{4})q_2+2\omega q_3-\tfrac{3}{4}\sqrt{3}\gamma q_1]\epsilon.
\end{aligned}\right\} \tag{18}$$

We now introduce into equations (18) the values chosen for the two constants of integration

$$a_1 = \epsilon, \qquad b_1 = 0, \tag{19}$$

and so get the two equations of condition

$$\omega^4 - \omega^2 + \tfrac{2}{1}\tfrac{7}{6} - \tfrac{2}{1}\tfrac{7}{6}\gamma^2 - (\omega^2 + \tfrac{9}{4})q_1 + \tfrac{3}{4}\sqrt{3}\gamma q_2 + 2\omega q_4 = 0, \tag{20}$$

$$\tfrac{3}{4}\sqrt{3}\gamma q_1 - (\omega^2 + \tfrac{3}{4})q_2 - 2\omega q_3 = 0. \tag{21}$$

Equation (20) corresponds to equation (6) and determines ω. As a first approximation we can drop the terms of the second order. The corresponding value of ω we shall call ω_0. We then have

$$\omega_0^4 - \omega_0^2 + \tfrac{2}{1}\tfrac{7}{6} - \tfrac{2}{1}\tfrac{7}{6}\gamma^2 = 0. \tag{22}$$

In the terms of the second order in equation (20) we can now substitute ω_0 for ω. The equation for determining ω then is

$$\omega^4 - \omega^2 + \tfrac{2}{1}\tfrac{7}{6} - \tfrac{2}{1}\tfrac{7}{6}\gamma^2 - (\omega_0^2 + \tfrac{9}{4})q_1 + \tfrac{3}{4}\sqrt{3}\gamma q_2 + 2\omega_0 q_4 = 0, \tag{23}$$

where ω_0 should be substituted for ω in the expressions for q_1, q_2 and q_4.

For the sake of brevity let us introduce a quantity E defined by the equation

$$E\epsilon^2 = (\omega_0^2 + \tfrac{9}{4})q_1 - \tfrac{3}{4}\sqrt{3}\gamma q_2 - 2\omega_0 q_4. \tag{24}$$

Introducing the expressions (17) for q_1, q_2 and q_4, substituting ω_0 for ω, we find

$$1152(5\omega_0^2 - 1)E = 496\omega_0^8 - 2908\omega_0^6 + 648\omega_0^4 + 2007\omega_0^2 - 243. \tag{25}$$

Equation (23) then reduces to the simple form

$$\omega^4 - \omega^2 + \tfrac{2}{1}\tfrac{7}{6} - \tfrac{2}{1}\tfrac{7}{6}\gamma^2 - E\epsilon^2 = 0. \tag{26}$$

From equations (22) and (26) we get

$$(\omega^2 + \omega_0^2 - 1)(\omega + \omega_0)(\omega - \omega_0) = E\epsilon^2. \tag{27}$$

It appears then that $\omega - \omega_0$ will in general be a small quantity of the second order. Provided that the first factors on the left-hand side of equation (27) do not vanish and provided E is a finite quantity we can substitute ω_0 for ω in the two first factors, getting

$$\omega = \omega_0 + \frac{E}{2\omega_0(2\omega_0^2 - 1)}\epsilon^2. \tag{28}$$

In the second equation of condition, equation (21), we can in the two last terms substitute ω_0 for ω. Introducing next the expressions for q_1, q_2 and q_3, substituting ω_0 for ω in these expressions also, we find making use of equation (22) that the second equation of condition is identically satisfied.

7. *Determination of the Terms of the Third Order in the Coefficients of the First Order a_{-1} and b_{-1}.*—Having determined ω we are now able to calculate the terms of the third order in a_{-1} and b_{-1}. The other two coefficients of the first order, a_1 and b_1, do not contain terms of order higher

than the first, as they have been fixed at specified values, being constants of integration.

The calculation of a_{-1} and b_{-1} is performed with the aid of the two first equations (16), into which we introduce $a_1 = \epsilon$ and $b_1 = 0$. We then get

$$\left.\begin{aligned} 2\omega a_{-1} &= \tfrac{3}{4}\sqrt{3}\gamma\epsilon - q_2\epsilon, \\ 2\omega b_{-1} &= -(\omega^2 + \tfrac{3}{4})\epsilon + q_1\epsilon. \end{aligned}\right\} \tag{29}$$

In the two terms of the third order, *i.e.* in $q_1\epsilon$ and $q_2\epsilon$, we can substitute ω_0 for ω. In the other terms, which are all of the first order, we are obliged to use the expression (28) for ω. After reduction we get

$$\left.\begin{aligned} a_{-1} &= \frac{3}{8\omega_0}\sqrt{3}\gamma\epsilon + \frac{1}{6144\omega_0{}^3(2\omega_0{}^2-1)(5\omega_0{}^2-1)}[-2880\omega_0{}^8 - 13496\omega_0{}^6 \\ &\qquad - 970\omega_0{}^4 + 4605\omega_0{}^2 - 783]\sqrt{3}\gamma\epsilon^3, \\ b_{-1} &= \frac{1}{8\omega_0}[-4\omega_0{}^2 - 3]\epsilon + \frac{1}{18432\omega_0{}^3(2\omega_0{}^2-1)(5\omega_0{}^2-1)}[50688\omega_0{}^{10} \\ &\qquad - 32736\omega_0{}^8 + 84512\omega_0{}^6 + 27466\omega_0{}^4 - 47175\omega_0{}^2 + 7209]\epsilon^3. \end{aligned}\right\} \tag{30}$$

We have now determined the fourteen coefficients in the Fourier series (3) retaining terms up to the third order.

8. *Determination of the Period.*—We shall now find an expression for the time of revolution of the periodic orbits. From equation (28) we get

$$\frac{1}{\omega} = \frac{1}{\omega_0} - \frac{E}{2\omega_0{}^3(2\omega_0{}^2-1)}\epsilon^2. \tag{31}$$

The period resulting from a calculation in which terms of the third order are retained we shall call T. The period resulting when terms of the first order only are retained is called T_0. From (31) then follows

$$T = T_0 - \frac{\pi E}{\omega_0{}^3(2\omega_0{}^2-1)}\epsilon^2. \tag{32}$$

Denoting the coefficient of ϵ^2 in equation (32) by K, we can write the equation as follows :

$$T = T_0 + K\epsilon^2, \tag{33}$$

where

$$\left.\begin{aligned} K = \frac{\pi}{1152\omega_0{}^3(2\omega_0{}^2-1)(5\omega_0{}^2-1)}[&-496\omega_0{}^8 + 2908\omega_0{}^6 - 648\omega_0{}^4 \\ &- 2007\omega_0{}^2 + 243]. \end{aligned}\right\} \tag{34}$$

Fig. 1 shows graphically the variation of K with $\omega_0{}^2$ for values of $\omega_0{}^2$ between the limits 0 and 1. From the diagram it is apparent that K is negative for values of $\omega_0{}^2$ between 0·5 and 1, *i.e.* for short-period orbits. We have thus found

The period of the short-period orbits around L_4 and L_5 decreases, as the orbit-size increases.

It is further apparent from the diagram that K is positive for values of

ω_0^2 between 0·2 and 0·5. Finally K is positive for $0 < \omega_0^2 < 0{\cdot}1188979$, while K is negative for $0{\cdot}1188979 < \omega_0^2 < 0{\cdot}2$.

The period of the long-period orbits around L_4 and L_5 for $0{\cdot}1188979 < \omega_0^2 < 0{\cdot}2$ decreases with increasing orbit-size. For values of ω_0^2 other than

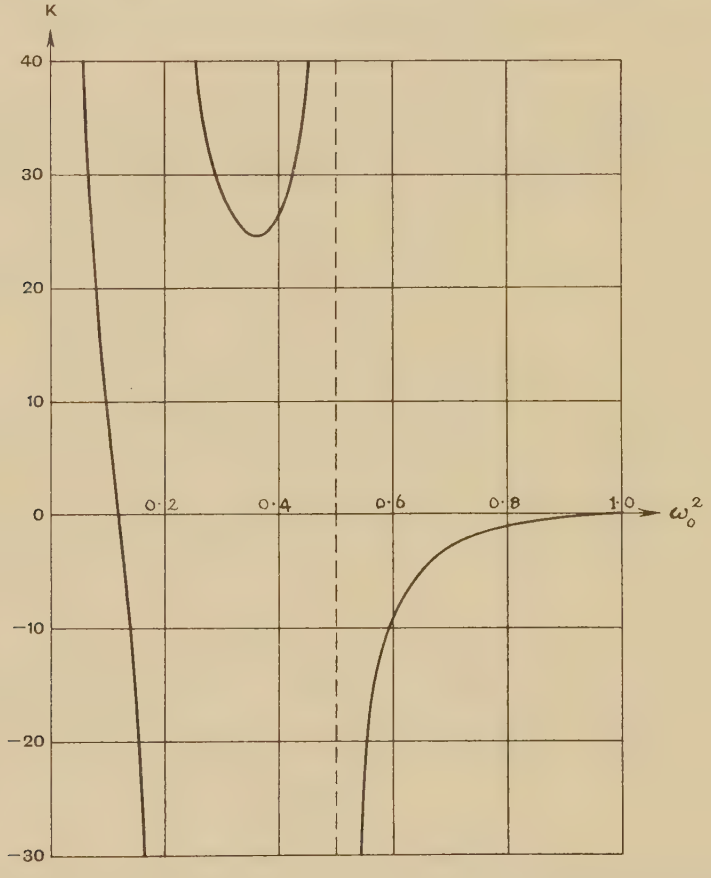

FIG. 1.—*The curve shows the variation of K with ω_0^2. As $T = T_0 + K\epsilon^2$ the value of K determines the variation of the period T of the periodic orbits around L_4 and L_5 with orbit-size ϵ.**

these the period of the long-period orbits will be increasing with increasing orbit-size.

The special case that ω_0^2 is close to $\frac{1}{2}$ was investigated in detail in my earlier paper. Here it was shown that for values of ω_0^2 a little greater than $\frac{1}{2}$, T will be a decreasing function of ϵ, while for values of ω_0^2 a little smaller than $\frac{1}{2}$, T will be an increasing function of ϵ. This result is in accord with the result here found, that the sign of K changes from $+$ to $-$, as ω_0^2 passes the value $\frac{1}{2}$.

Figs. 2, 3 and 4 show the shape and the orientation with respect to the

* It should be emphasised, however, that there are in the interval $0 < \omega_0^2 < 0{\cdot}2$ an infinite number of values of ω_0^2, for which no periodic orbits exist.

co-ordinate-system of four periodic orbits, the value of the orbit constant ϵ having been chosen equal to 0·2.

Fig. 2 shows two short-period orbits (ω_0^2 equal to 1 and 0·6). The orbit extending the farthest to the left corresponds to $\omega_0^2 = 0{\cdot}6$. From the diagram it is apparent that the shape of the orbit is only little changed, when ω_0^2 decreases from 1 to 0·6.

Fig. 3 shows a short-period orbit ($\omega_0^2 = 0{\cdot}6$) and the corresponding long-period orbit ($\omega_0^2 = 0{\cdot}4$). The orbit extending the farthest to the left

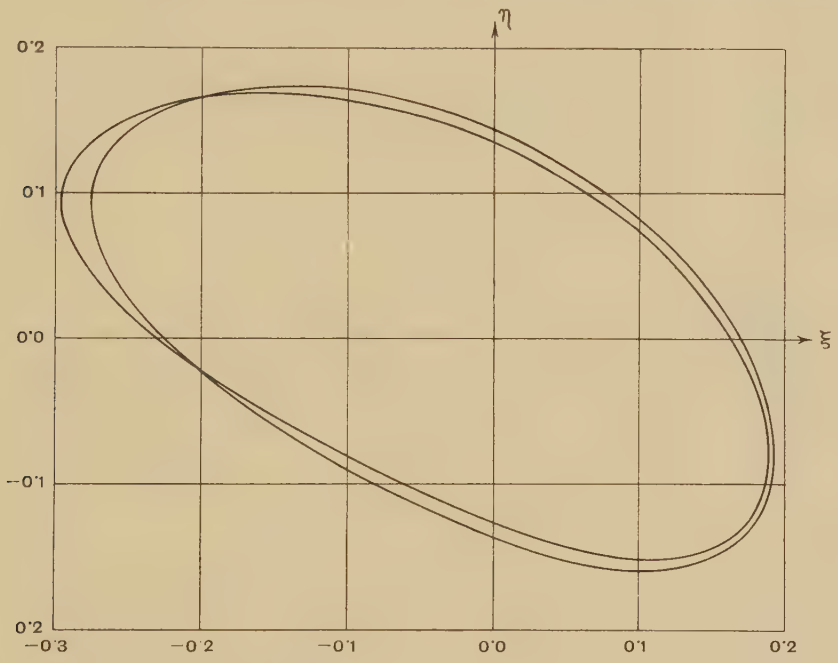

FIG. 2.—*Periodic orbits around L_4. The orbit constant ϵ is equal to 0·2 for both orbits, while ω_0^2 is equal to 1 and 0·6 respectively.*

corresponds to $\omega_0^2 = 0{\cdot}4$. It is seen that there is a rather great change of shape as ω_0^2 decreases from 0·6 to 0·4, the orbit becoming more oblong.

Finally in fig. 4 two long-period orbits are shown (ω_0^2 equal to 0·4 and 0·3). The orbit extending the farthest to the left corresponds to $\omega_0^2 = 0{\cdot}3$. As ω_0^2 decreases from 0·4 to 0·3, the orbit gets still more oblong and at the same time displaced to the left.

9. *Critical Values of ω_0.*—It appears from equations (14) that it is not possible to determine the coefficients of the second order a_2, b_2, a_{-2} and b_{-2}, when ω_0^2 is equal to $\frac{1}{5}$. Similarly it is not possible to determine the coefficients of the third order a_3, b_3, a_{-3} and b_{-3}, when ω_0^2 is equal to $\frac{1}{10}$. When ω_0^2 assumes one of these critical values, the determinant of the four equations determining the coefficients vanishes.

We shall now try to determine all critical values of ω_0^2. It will be natural to separate the investigation into three parts.

1°. *The Determination of the Coefficients of the First Order.*—Having

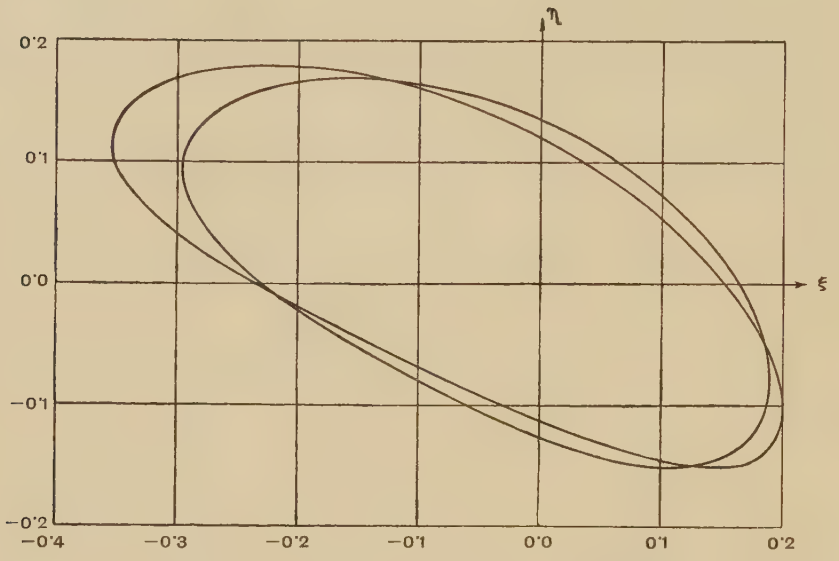

FIG. 3.—*Periodic orbits around L_4. The orbit constant ϵ is equal to 0·2 for both orbits, while ω_0^2 is equal to 0·6 and 0·4 respectively.*

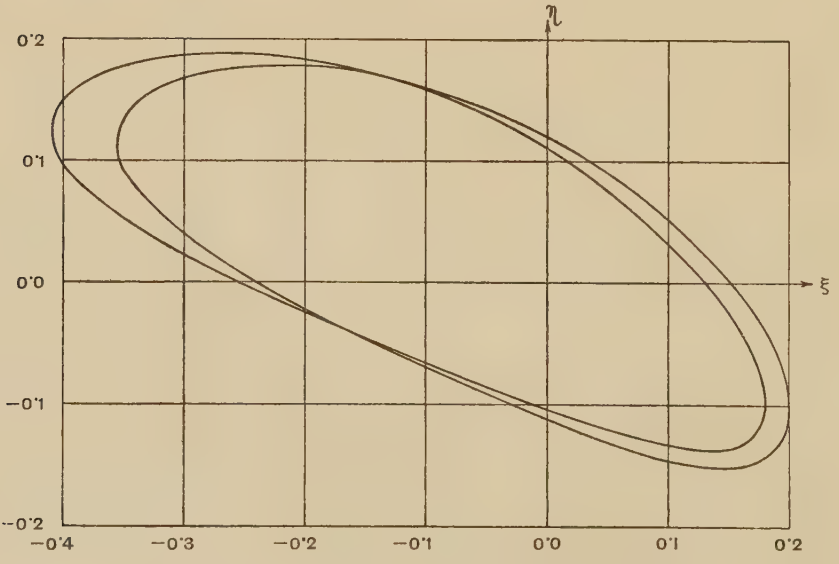

FIG. 4.—*Periodic orbits around L_4. The orbit constant ϵ is equal to 0·2 for both orbits, while ω_0^2 is equal to 0·4 and 0·3 respectively.*

chosen the values ϵ and 0 for the constants of integration a_1 and b_1, we determine the two other coefficients a_{-1} and b_{-1} with the aid of the two first equations (4). Apparently a_{-1} and b_{-1} cannot be determined when $\omega = 0$. A closer examination leads, however, to a different result.

Putting $\omega = 0$ in the equations (4) we get the following four equations

$$\left.\begin{aligned} \tfrac{3}{4}a_1 + \tfrac{3}{4}\sqrt{3}b_1 &= 0, \\ \tfrac{3}{4}\sqrt{3}a_1 + \tfrac{9}{4}b_1 &= 0, \\ \tfrac{3}{4}a_{-1} + \tfrac{3}{4}\sqrt{3}b_{-1} &= 0, \\ \tfrac{3}{4}\sqrt{3}a_{-1} + \tfrac{9}{4}b_{-1} &= 0. \end{aligned}\right\} \tag{35}$$

The first and the second of these equations are identical, and also the third and the fourth. The four coefficients then only have to satisfy the two equations

$$\left.\begin{aligned} \tfrac{3}{4}a_1 + \tfrac{3}{4}\sqrt{3}b_1 &= 0, \\ \tfrac{3}{4}a_{-1} + \tfrac{3}{4}\sqrt{3}b_{-1} &= 0. \end{aligned}\right\} \tag{36}$$

From the first equation (36) it is seen that a_1 and b_1 in this special case are not independent and therefore cannot both be chosen to be constants of integration. Keeping b_1 as constant of integration we can choose either a_{-1} or b_{-1} as the other constant of integration. The two other coefficients are then determined by equations (36).

Let us choose

$$\left.\begin{aligned} b_1 &= 0, \\ b_{-1} &= \sigma, \end{aligned}\right\} \tag{37}$$

where as before b_1 is chosen equal to zero, while σ is a new orbit constant. The two other coefficients are found from equations (36):

$$\left.\begin{aligned} a_1 &= 0, \\ a_{-1} &= -\sqrt{3}\sigma. \end{aligned}\right\} \tag{38}$$

On the other hand, introducing (37) in (4) we find

$$\left.\begin{aligned} a_1 &= -\frac{8\omega}{4\omega^2 + 3}\sigma, \\ a_{-1} &= -\frac{3\sqrt{3}\gamma}{4\omega^2 + 3}\sigma, \end{aligned}\right\} \tag{39}$$

which shows that $\omega = 0$ is not a critical value in the determination of the coefficients of the first order, when the integration constants are properly chosen.

As the values of a_{-1} and b_{-1} given in equations (10) are used in the determination of the other coefficients of the Fourier series, ω will appear as a divisor in the expressions for these coefficients. If, however, the constants of integration (37) are used instead of $a_1 = \epsilon$, $b_1 = 0$, then ω will not appear as a divisor, and $\omega = 0$ will not be a critical value in this respect.

2°. *The Determination of the Constant Terms a_0 and b_0.*—The determinant of the two equations from which a_0 and b_0 are found is

$$D_0 = \tfrac{2}{1}\tfrac{7}{6}(1-\gamma^2). \tag{40}$$

The determinant vanishes for $\gamma = 1$, *i.e.* for

$$\omega_0^2 = \begin{cases} 0 \\ 1. \end{cases} \tag{41}$$

The two equations determining a_0 and b_0 are in both cases identical, and the equations are satisfied for an infinite number of sets of values of a_0 and b_0. Therefore $\omega_0^2 = 0$ and $\omega_0^2 = 1$ are critical values in the determination of a_0 and b_0.

3°. *The Determination of the Coefficients of Orders Higher than the First.*—The four equations for the determination of the coefficients of the nth order take the form

$$\left.\begin{aligned}
(n^2\omega_0^2 + \tfrac{3}{4})a_n \quad &+ \tfrac{3}{4}\sqrt{3}\gamma b_n \quad &&+ 2n\omega_0 b_{-n} = s_{n,1}, \\
\tfrac{3}{4}\sqrt{3}\gamma a_n + (n^2\omega_0^2 + \tfrac{9}{4})b_n \quad &- 2n\omega_0 a_{-n} \quad &&= s_{n,2}, \\
- 2n\omega_0 b_n + (n^2\omega_0^2 + \tfrac{3}{4})a_{-n} \quad &&&+ \tfrac{3}{4}\sqrt{3}\gamma b_{-n} = s_{n,3}, \\
2n\omega_0 a_n \quad &&&+ \tfrac{3}{4}\sqrt{3}\gamma a_{-n} + (n^2\omega_0^2 + \tfrac{9}{4})b_{-n} = s_{n,4},
\end{aligned}\right\} \tag{42}$$

where $s_{n,1}$, $s_{n,2}$, $s_{n,3}$ and $s_{n,4}$ are quantities of the nth order. Denoting the determinant corresponding to these four equations by D_n, one finds

$$D_n = [n^4\omega_0^4 - n^2\omega_0^2 + \tfrac{2}{1}\tfrac{7}{6}(1-\gamma^2)]^2. \tag{43}$$

As ω_0^2 has to satisfy equation (22) we can transform the expression for D_n as follows :

$$D_n = (n^2-1)^2\omega_0^4[(n^2+1)\omega_0^2 - 1]^2. \tag{44}$$

As we have assumed that $n \geqq 2$, it is seen that D_n vanishes only when

$$\omega_0^2 = \begin{cases} 0 \\ 1 \\ \dfrac{}{n^2+1}, \end{cases} \qquad (n \geqq 2). \tag{45}$$

The values given in (45) are critical values of ω_0^2.

We can sum up the result of the preceding discussion as follows : In the determination of the Fourier series we encounter the following critical values :—

(i) $\omega_0^2 = 0$ is a critical value in the determination of the constant terms and in the determination of all terms of orders higher than the first.

(ii) $\omega_0^2 = 1$ is a critical value in the determination of the constant terms.

(iii) $\omega_0^2 = \dfrac{1}{n^2+1}$ is a critical value in the determination of the terms of the nth order ($n \geqq 2$).

10. *The Critical Long-period Solutions.*—A closer investigation of the critical values

$$\omega_0^2 = \frac{1}{n^2+1}, \qquad (n \geqq 2) \tag{46}$$

is of particular interest.

With $n=2$ the critical value is $\frac{1}{5}$, with $n=3$ it is $\frac{1}{10}$, viz. just the values which are found in calculating the coefficients of the second and third order from equations (14) and (15).

As the values in (46) are smaller than $\frac{1}{2}$, they all correspond to long-period solutions. For the corresponding short-period solutions we get

$$\omega_0^2 = \frac{n^2}{n^2+1}, \qquad (n \geqq 2). \tag{47}$$

Denoting by T_l the period corresponding to the critical values stated in (46) and denoting by T_s the period for the corresponding short-period orbits, we have

$$T_l^2 = 4\pi^2(n^2+1), \qquad T_s^2 = 4\pi^2\frac{n^2+1}{n^2}, \tag{48}$$

from which it appears that

$$T_l = nT_s, \qquad (n \geqq 2). \tag{49}$$

We have thus found :

The critical long-period solutions have the property that the period is a multiple of the period in the corresponding non-critical short-period solutions.

11. *Summary.*—Fourier expansions of the periodic orbits in the immediate vicinity of the triangular libration points in the restricted problem of three bodies are obtained, valid for all values of the finite masses, for which such orbits exist. Terms up to the third order with regard to the orbital dimensions have been retained. The period of the periodic orbits is determined as a function of the linear dimensions, which has the form $T = T_0 + K\epsilon^2$, where T is the period and ϵ measures the linear orbital dimension. K is a function of T_0, which in its turn is a function of the ratio of the two finite masses. A diagram showing the variation of K is given. Four periodic orbits are shown in graphical form also.

It turns out that the determination of the coefficients of the Fourier series is impossible for certain special values of T_0. This question is investigated in a general way. The question has been investigated earlier by other methods by Buck, Greaves and Wintner. It is shown that the short-period orbits always exist, while the long-period orbits exist, except when the ratio of the periods of the long-period and the short-period libration ellipses is an integer.

Viborg, Denmark :
1935 *February.*

ERROR IN A RECENT PAPER ON PROPER MOTIONS.

H. Knox-Shaw.

In a paper entitled "An Analysis of the Radcliffe Proper Motions in the Selected Areas," published a year ago,* I gave some positions of the Solar Apex and Vertex derived from photographic proper motions determined at the Radcliffe Observatory. The probable errors of these positions all contain an error of calculation. They should be multiplied by $10/\sqrt{n}$, where n is twice the number of regions entering into the solutions and varies from 50 to 60. The value of this factor for each case is given below.

Page			Factor
402	Solar Apex	Brighter Stars	1·31
	,, ,,	Fainter Stars	1·34
403	Vertex	Brighter Stars	1·29
	,,	Fainter Stars	1·34
404	Solar Apex	...	1·41
	Vertex	...	1·34

The general conclusions of that paper are not affected by this error.

The same error enters into all the probable errors in the chapter headed "The Solar Apex and Vertex of Star Streaming" in the *Radcliffe Catalogue of Proper Motions*, but into no other part of that volume.

I am indebted to Professor S. A. Mitchell for drawing my attention to this error. Drs. van de Kamp and Vyssotsky discovered it in comparing the Radcliffe determinations of the Solar Apex and Vertex with those given by some Leander McCormick proper motions which they are preparing for publication.

Radcliffe Observatory, Oxford :
1935 *February* 23.

OBSERVATIONS OF OCCULTATIONS MADE AT TEDDINGTON DURING 1934.

John L. Haughton, D.Sc., F.Inst.P.

The observations up to, and including, No. 38 were made as described previously (*M.N.*, **94**, 484, 1934). After that the telescope was moved from the National Physical Laboratory to an observatory built in my garden. The co-ordinates, deduced from a 1-inch Ordnance Survey map, are

* *M.N.*, **94**, 399, 1934.

$\phi = 5^\circ\ 25'\ 6''$ N.
$\lambda = 0^\circ\ 19'\ 3''$ W.

The time was obtained from a Synchronome clock built from castings kindly supplied by Mr. Hope-Jones of The Synchronome Company. This kept much better time than the clock used last year, the extreme divergence from the mean rate being ± 0·5 second per month. Observations Nos. 40 to 43 were made before the clock was properly regulated and are, therefore, slightly doubtful. The probable errors given in the table for these occultations are deduced on the assumption that the clock correction is right.

No.	Date	Star	Mag.	Phase	G.C.T.	Probable Error. Seconds	Remarks
					h m s		
28	Feb. 1	48 Leonis	5·2	R	20 14 11·7	?	Very doubtful.
29	20	7 Tauri	5·0	D	19 19 59·4	± 0·5	See below.
30	Mar. 19	W.Z.C. 189	6·9	D	19 02 45·8	± 2·0	Haze. Very difficult.
35	27	A Leonis	4·0	D	20 32 05·4	± 0·5	D very sharp.
37	May 17	48 Geminorum	5·8	D	22 07 20·8	± 1·0	
38	22	p^4 Leonis	5·6	D	23 19 52·2	± 1·5	
40	Aug. 31	17 Tauri	3·8	D	00 58 26·2	± 4·0	Doubtful.
41	31	q Tauri	4·3	D	01 18 35·3	± 3·0	A difficult observation. See below.
42	31	20 Tauri	4·1	D	01 26 33·0	± 1·0	
43	31	17 Tauri	3·8	R	01 47 22·0	± 1·0	R very sharp.
47	Oct. 24	27 Tauri	3·7	R	21 45 9·2	± 1·5	Thin clouds. R very sharp.
48	24	28 Tauri	5·2	R	21 49 52·8	± 2·5	Not a good observation. See below.
49	26	125 Tauri	5·1	R	20 33 58·5	± 1·0	
50	28	ϵ Geminorum	3·2	R	00 47 28·5	?	See below.

Notes

No. 29. Star dimmed very definitely (by about 20 per cent.) approximately 0·5 to 1·0 second before D.

No. 41. D very difficult to observe on fringe of limb. Probably 1 or 2 seconds early.

No. 48. Star may have reappeared 1 or 2 seconds earlier.

No. 50. Completely clouded at D. Clouds drifting past at time of R. Star first seen at time given in table, and suddenness of apparition suggests that this may be the correct time of R. In any case, it is unlikely to be as much as 10 seconds earlier.

1935 *February* 5.

OCCULTATIONS OF STARS BY THE MOON, OBSERVED AT THE TOMSK UNIVERSITY OBSERVATORY DURING THE YEAR 1934.

N. Goryatscheff.

Date (G.C.T.)	Star	Mag.	Phase	Local Sidereal Time	Δt
				h m s	s
V 17·689789	B.D. +24° 1491	8·4	D	13 51 48·9	+10·2
V 17·723282	ω Geminorum	5·2	D	14 40 10·6	+ 7·9
VIII 3·797816	20 Tauri	4·1	R	21 35 19·5	+ 6·4
VIII 3·830251	B.D. +23° 540	6·8	R	22 22 09·5	+ 5·4
VIII 3·849780	B.D. +23° 553	6·6	R	22 50 21·5	+ 2·4
VIII 3·857837	B.D. +23° 562	7·8	R	23 01 59·5	+ 5·3
VIII 28·665431	101 Piscium	6·2	R	20 02 43·9	+ 6·4

Apparent Places of Occulted Stars

	h m s	° ′ ″	
B.D. +24° 1491	6 56 52·41	+24 34 32·1	Gr_{1910} 3515
ω Geminorum	6 58 24·82	+24 18 45·1	Am. Ephem.
20 Tauri	3 41 56·29	+24 09 58·3	Am. Ephem.
B.D. +23° 540	3 43 36·09	+24 05 20·9	Gr_{1910} 1669. Abbadia 1719
B.D. +23° 553	3 44 36·48	+24 08 50·9	Gr_{1910} 1681. Abbadia 1732
B.D. +23° 562	3 45 32·45	+24 03 03·5	Abbadia 1746
101 Piscium	1 32 18·27	+14 19 52·1	Am. Ephem.

Instrument : Zeiss telescope of 90-mm. aperture.

Observations with eye and ear.

Observers : for 101 Piscium, Mr. Suschareff ; for all others, Prof. N. Goryatscheff.

The proper motions of stars are computed from the Greenwich 1910 Catalogue and from Schorr's *Eigenbewegungslexikon.*

Tomsk University Observatory :
1935 *January* 13.

OCCULTATIONS OF STARS BY THE MOON, OBSERVED AT THE ASTRONOMICAL OBSERVATORY, KIEW, FROM 1934 AUGUST TO 1935 JANUARY.

S. D. Tscherny.

No.	Date	G.C.T.	Star	Mag.	Dis- or Re-ap.	Telescope and Power	Observer
	1934	h m s					
1	Aug. 21	20 1 56·17	ψ Sagittarii	4·8	D	U_{74}	I
2	Sept. 16	18 24 59·73	χ Sagittarii	4·4	D	R_{120}	T
3	16	18 24 59·76	χ Sagittarii	4·4	D	U_{74}	I
4	16	18 24 59·29	χ Sagittarii	4·4	D	B_{61}	R
5	19	20 26 8·67	19 Capricorni	5·7	D	R_{120}	T
6	19	20 26 9·74	19 Capricorni	5·7	D	U_{74}	I
7	19	20 26 8·95	19 Capricorni	5·7	D	B_{61}	R
8	20	18 53 10·95	151 B Capricorni	6·1	D	R_{120}	T
9	20	18 53 10·85	151 B Capricorni	6·1	D	U_{74}	I
10	Oct. 24	19 11 9·87	17 Tauri	3·8	D	R_{120}	T
11	24	19 35 49·72	23 Tauri	4·3	D	R_{120}	T
12	24	19 58 13·14	17 Tauri	3·8	R	R_{120}	T
13	24	20 12 10·85	η Tauri	2·9	D	R_{120}	T
14	24	20 39 6·42	23 Tauri	4·3	R	R_{120}	T
15	24	21 6 15·96	28 Tauri	5·2	D	R_{120}	T
16	24	21 10 15·31	27 Tauri	3·7	D	R_{120}	T
17	24	21 22 7·36	η Tauri	2·9	R	R_{120}	T
18	26	19 48 18·04	125 Tauri	5·1	D	R_{120}	T
19	26	20 18 15·12	125 Tauri	5·1	R	R_{120}	T

Instruments used :

R = 243 mm. Merz-Repsold refractor.
B = 83 mm. Busch refractor.
U = 72 mm. Utzschneider and Fraunhofer refractor.

Observers :

T = S. Tscherny.
I = I. Iljinsky.
R = N. Rudsky.

Notes

Nos. 3, 5, 8, 9, 19 : instantaneous.
Nos. 11, 13, 16, 17 : very good.
Nos. 2, 4, 10, 12, 14, 15, 18 : good.
No. 7 : satisfactory.
Nos. 1, 6 : doubtful.

Astronomical Observatory, Kiew :
1935 *February* 2.

Erratum

Council Note on Variable Stars, *M.N.* 95, 407, line 18 from top, *for* J.B.A.A. *read* M.B.A.A.